Some results in the theory of formations of finite groups

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Introduction

1. A formation is a class of groups $\mathfrak{F}$ which is closed under taking homomorphic images and subdirect products. In particular, every finite group G has a smallest normal subgroup with quotient in $\mathfrak{F}$ called the $\mathfrak{F}$-residual of G and denoted by $G^{\mathfrak{F}}$.
2. $\mathfrak{F}$ is subgroup-closed if it is closed under taking subgroups, that is, $U^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ for all subgroups U of G.
3. A Fitting class is a class of groups $\mathfrak{F}$ which is closed under taking subnormal subgroups and such that the subgroup $G_{\mathfrak{F}}$, generated by the subnormal $\mathfrak{F}$-subgroups of a group G, is itself an $\mathfrak{F}$-group. This subgroup is called the $\mathfrak{F}$-radical of G.
4. A formation $\mathfrak{F}$ is saturated (respectively solubly saturated) if a group G belongs to $\mathfrak{F}$ if the Frattini factor group $G/\Phi(G)$ (respectively $G/\Phi(G_{\mathfrak{S}})$) belongs to $\mathfrak{F}$.

Introduction

Let $\mathfrak{F}$ be a formation. A maximal subgroup M of a group G is said to be $\mathfrak{F}$-normal in G if the primitive group $G/\mathrm{Core}_G(M)$ belongs to $\mathfrak{F}$. It is clear that M is $\mathfrak{F}$-normal in G if and only if M contains $G^{\mathfrak{F}}$.

Introduction

Definition

Let $\mathfrak{F}$ be a formation. A subgroup U of a group G is called an $\mathfrak{F}$-subnormal subgroup of G if either $U = G$ or there is a chain of subgroups

$$U = U_0 < U_1 < \cdots < U_n = G$$

such that U_{i-1} is a maximal $\mathfrak{F}$-normal subgroup of U_i, for $i = 1, 2, \ldots, n$.

It is rather clear that the $\mathfrak{N}$-subnormal subgroups of a group G for the formation $\mathfrak{N}$ of all nilpotent groups are subnormal, and they coincide in the soluble universe.

Wielandt's properties

Theorem (Wielandt's join theorem)

The subgroup generated by two subnormal subgroups of a group G is itself subnormal in G.

As a result, the set $sn(G)$ of all subnormal subgroups of a group G is a sublattice of the subgroup lattice of G.

Wielandt's properties

Let $\mathfrak{F}$ be a formation. One might wonder whether the set of $\mathfrak{F}$-subnormal subgroups of a group forms a sublattice of the subgroup lattice. The answer is in general negative. The formation of all 2-nilpotent groups and the group $G = [V]X$, where X is the symmetric group of degree 4 and V is an irreducible and faithful module over the field of 3-elements provide a counterexample.

Wielandt's properties

The classification problem of the lattice formations, that is, formations $\mathfrak{F}$ for which the set of $\mathfrak{F}$-subnormal subgroups is a sublattice of the subgroup lattice was proposed by Shemetkov in 1978 and it appeared in the Kourovka Notebook in 1984 as Problem 9.75.

Wielandt's properties

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- A. F. Vasil'ev, S. F. Kamornikov, V. Semenchuk. On lattices of subgroups of finite groups. Infinite groups and related algebraic structures, Institut Matematiki AN Ukrainy, Kiev, (1993), 27–54.

Wielandt's properties

Theorem

Let $\mathfrak{F}$ be a saturated formation. Then $\mathfrak{F}$ is a lattice formation if and only if $\mathfrak{F} = \mathfrak{M} \times \mathfrak{H}$ for some subgroup-closed saturated formations $\mathfrak{M}$ and $\mathfrak{H}$ satisfying the following conditions:

1. $\pi(\mathfrak{M}) \cap \pi(\mathfrak{H}) = \emptyset$.
2. *There exists a set of prime numbers π^* and a partition $\{\pi_i : i \in \mathcal{I}\}$ of π^* such that $\mathfrak{H} = \times_{i \in \mathcal{I}} \mathfrak{S}_{\pi_i}$.*
3. $\mathfrak{M} = \mathfrak{S}_p \mathfrak{M}$ *for all $p \in \pi(\mathfrak{M})$ and $\mathfrak{M}$ is an $\mathfrak{M}^2$-normal Fitting class.*
4. *Every non-cyclic $\mathfrak{M}$-critical group G with $\Phi(G) = 1$ is a primitive group of type 2 such that $G/\operatorname{Soc}(G)$ is a cyclic group of prime power order.*

Wielandt's properties

Theorem (Wielandt's property for nilpotent residuals)

The nilpotent residual of the subgroup generated by two subgroups is the subgroup generated by the nilpotent residuals of the subgroups.

As a consequence $\mathrm{F}(K)$, the Fitting subgroup of a subnormal subgroup K of G, normalises the nilpotent residual of every subnormal subgroup of G.

Wielandt's properties

For a group G and the lattice $\mathrm{S_n}(G)$ of all subnormal subgroups of G, a map $\omega\colon\ \mathrm{S_n}(G) \longrightarrow \mathrm{S_n}(G)$ is called a Wielandt operator in G if, for any $H, K \in \mathrm{S_n}(G)$, the following conditions are satisfied:

W1: $\langle H, K\rangle^{\omega} = \langle H^{\omega}, K^{\omega}\rangle$,

W2: if $H \trianglelefteq K$, then $H^{\omega} \trianglelefteq K$.

Here, of course, H^{ω} denotes the image of H under the map ω. Note that Condition W2 implies that H^{ω} is a normal subgroup of H.

Wielandt's properties

Theorem (Wielandt)

Let φ and ψ be two Wielandt operators in a group G. Assume that two subnormal subgroups H and K of G are permutable if $H = H^{\varphi} = H^{\psi}$. Then $A^{\varphi}B^{\psi} = B^{\psi}A^{\varphi}$ for any pair (A, B) of subnormal subgroups of G.

Wielandt's properties

Suppose that a Wielandt operator ω is defined in all groups G. If ω satisfies $(X^{\omega})^{\alpha} = (X^{\alpha})^{\omega}$ for any homomorphism α of a group X, then the class $\mathfrak{F} := (X \mid X^{\omega} = 1)$ is a Fitting formation and G^{ω} is the $\mathfrak{F}$-residual of G for every group G. Conversely if $\mathfrak{F}$ is a Fitting formation, then the map $\delta\colon\ \mathrm{S_n}(G) \longrightarrow \mathrm{S_n}(G)$, $H^{\delta} = H^{\mathfrak{F}}$ for all $H \in \mathrm{S_n}(G)$, defines a Wielandt operator in every group G, permuting with all homomorphisms provided that δ satisfies Condition W1.

Wielandt's properties

Consequently, the problem of finding Wielandt operators which are permutable with homomorphisms is reduced to the description of Fitting formations $\mathfrak{F}$ satisfying the following property:

If U and V are subnormal subgroups of a group G, then $\langle U, V\rangle^{\mathfrak{F}} = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$.

Wielandt's properties

Definition

Let $\mathfrak{F}$ be a formation. We say that $\mathfrak{F}$ satisfies the Wielandt property for residuals if whenever U and V are subnormal subgroups of $\langle U, V\rangle$ in a group G, then $\langle U, V\rangle^{\mathfrak{F}} = \langle U^{\mathfrak{F}}, V^{\mathfrak{F}}\rangle$.

Wielandt's properties

- S. F. Kamornikov, L. A. Shemetkov. On coradicals of subnormal subgroups. Algebra i Logika, **34** (1995), 493–513.
- A. Ballester-Bolinches, John Cossey, L. M. Ezquerro. On formations of finite groups with the Wielandt property for residuals. J. Algebra, **243** (2001), 717–737.

Wielandt's properties

- Every soluble subgroup-closed Fitting formation satisfies the Wielandt property for residuals.
- Some Fitting formations defined by a Fitting family of modules (in the sense of Cossey and Kanes) satisfies the Wielandt property for residuals.
- For solubly saturated Fitting formations, the problem can be reduced to the boundary.

Wielandt's properties

Theorem

Let $\mathfrak{F}$ be a Fitting formation. If U and V are subgroups of a group G such that U and V are subnormal in $\langle U, V\rangle$, it follows that $U_{\mathfrak{F}}$ normalises $V^{\mathfrak{F}}$. In particular, the $\mathfrak{F}$-radical of G normalises the $\mathfrak{F}$-residual of every subnormal subgroup of G.

Wielandt's properties

Definition

Let $\mathfrak{F}$ be a non-empty formation. We say that $\mathfrak{F}$ has the generalised Wielandt property for residuals, $\mathfrak{F}$ is a GWP-formation for short, if $\mathfrak{F}$ enjoys the following property: If G is a group generated by two $\mathfrak{F}$-subnormal subgroups A and B, then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$.

Wielandt's properties

- S. F. Kamornikov. Permutability of subgroups and $\mathfrak{F}$-subnormality. Siberian Math. J., **37** (1996), 936–949.
- A. Ballester-Bolinches, M. C. Pedraza-Aguilera, M. D. Pérez-Ramos. On $\mathfrak{F}$-subnormal subgroups and $\mathfrak{F}$-residuals of finite groups. J. Algebra, **186** (1996), 314–322.
- A. Ballester-Bolinches. $\mathfrak{F}$-critical groups, $\mathfrak{F}$-subnormal subgroups, and the generalised Wielandt property for residuals. Ric. Mat., **186**, (2006), 13–30.

Wielandt's properties

Theorem

Every GWP-formation $\mathfrak{F}$ is a subgroup-closed Fitting formation for which the set of all $\mathfrak{F}$-subnormal subgroups of every group G is a sublattice of the subgroup lattice of G, that is, $\mathfrak{F}$ is a lattice formation.

Wielandt's properties

Theorem (B-B, Ric. Mat., 2006)

Let $\mathfrak{F}$ be a subgroup-closed saturated lattice formation. Then $\mathfrak{F}$ is a GWP-formation if and only if there exists a subclass $\mathfrak{Y}$ of $b_n(\mathfrak{F})$ such that the following condition is fulfilled by all groups $G \in \mathfrak{Y}$: If $G = \langle A, B \rangle$ with A and B $\mathfrak{F}$-subnormal subgroups of G, then $G^{\mathfrak{F}} = \langle A^{\mathfrak{F}}, B^{\mathfrak{F}} \rangle$.

Wielandt's properties

A. Ballester-Bolinches, S. F. Kamornikov, V. Pérez-Calabuig. On formations of finite groups with the generalised Wielandt property for residuals. J. Algebra, **412** (2014), 173–178.

Wielandt's properties

Theorem

Every GWP-formation is saturated

Wielandt's properties

Definition

Let $\mathfrak{X}$ be a class of groups. A group G is said to be $\mathfrak{X}$-critical (or critical for $\mathfrak{X}$) if $G \notin \mathfrak{X}$, but all proper subgroups of G belong to $\mathfrak{X}$.

Wielandt's properties

Theorem

Let $\mathfrak{F}$ be a GWP-formation, and G an $\mathfrak{F}$-critical group. Then $N/\Phi(G) = G^{\mathfrak{F}}\Phi(G)/\Phi(G) = \mathrm{Soc}(G/\Phi(G))$ is a minimal normal subgroup of $G/\Phi(G)$. If N is a proper subgroup of G, then G/N is a cyclic q-group for some prime $q \in \pi(\mathfrak{F})$ and $N/\Phi(G)$ is a q'-group if $N/\Phi(G)$ is abelian.

It follows that a GWP-formation must be solubly saturated. This is the first step to proof the saturation theorem.

Wielandt's properties

Theorem (Kamornikov, B-B, submitted)

Let $\mathfrak{M}$ be a subgroup-closed extensible formation and $\mathfrak{X}$ be a class of simple non-abelian $\mathfrak{M}$-critical groups. Set $\mathfrak{L}$ $=\mathfrak{M}$form$(\mathfrak{X})$. Assume that $\mathfrak{H}$ is a formation such that $\pi(\mathfrak{L}) \cap \pi(\mathfrak{H}) = \emptyset$. If $\mathfrak{F} = \mathfrak{L} \times \mathfrak{H}$ and there exists a partition $\{\pi_i : i \in I\}$ of $\pi(\mathfrak{H})$ such that $\mathfrak{H}=\times_{i\in I}\mathfrak{S}_{\pi_i}$, then $\mathfrak{F}$ is a GWP-formation.

As a consequence, every soluble formation is a GWP-formation if and only if it is a lattice formation.