Free algebras generated by symmetric elements inside division rings with involution

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joint with Vitor O. Ferreira and Jairo Z. Gonçalves

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- Vitor O. Ferreira, Jairo Z. Gonçalves and J. S., *Free symmetric group algebras in division rings generated by poly-orderable groups*, J. Algebra, **392** (2013), 69–84.

- Vitor O. Ferreira, Jairo Z. Gonçalves and J. S., *Free symmetric algebras in division rings generated by enveloping algebras of Lie algebras*, arXiv:1406.3078 (2014).

Notation

- *Rings* and *algebras* are associative with 1.
- Morphisms, subrings, subalgebras and embeddings of these objects preserve 1.
- We also use *Lie algebras* and morphisms of Lie algebras.
- A domain is a nonzero ring that contains no zero divisors other than zero.
- A division ring or skew field is a nonzero ring such that every nonzero element is invertible.
- Free algebras $\Bbbk\langle X\rangle$ are supposed to be noncommutative, i.e. $|X| \geq 2$.
  $\Bbbk\langle X\rangle$ is the set of polynomials where $xy \neq yx$ if $x, y \in X$, $x \neq y$.
  For example, $x^2y \neq yx^2 \neq xyx$.

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A conjecture and a question

Conjecture A (Makar-Limanov)

Let D be a division ring with center Z.

(A) If D is finitely generated (as a division ring) over Z and $[D : Z] = \infty$, then D contains a free Z-algebra

- If $\mathbb{k} < Z$, D contains free Z algebras $\Leftrightarrow$ D contains free $\mathbb{k}-$algebras.
- If $[D : Z] = n < \infty$, then $D \hookrightarrow End_Z(D) = M_n(Z)$. The ring $M_n(Z)$ is P.I.
- $A_1 = \langle x, y \mid yx - xy = 1\rangle$ is of polynomial growth, but its Ore division ring of fractions D_1 contains a free algebra.

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- $A_1 = \langle x, y \mid yx - xy = 1\rangle$ is of polynomial growth, but its Ore division ring of fractions D_1 contains a free algebra.

Let $\mathbb{k}$ be a field and A be a $\mathbb{k}$-algebra.

- $\mathbb{k}$-involution on A is a $\mathbb{k}$-linear map $^\star\colon A \to A$ satisfying

$$(ab)^\star = b^\star a^\star, \quad \forall a, b \in A, \qquad \text{and} \qquad (a^\star)^\star = a, \quad \forall a \in A.$$

- An element $a \in A$ is said to be symmetric if $a^\star = a$.

Question

Let D be a division $\mathbb{k}$-algebra with a $\mathbb{k}$-involution $\star\colon D \to D$.

(SA) If D satisfies conjecture (A), does D contain a free $\mathbb{k}$-algebra generated by symmetric elements?

Definition

Let $\Bbbk$ be a field and G a group. A group algebra is a ring:

- As a set $\Bbbk[G] = \{\sum_{x \in G} x a_x \mid a_x \in \Bbbk \text{ almost all } a_x = 0\}$.
- Sum:
$$\sum_{x \in G} x a_x + \sum_{x \in G} x b_x = \sum_{x \in G} x(a_x + b_x)$$
- Multiplication:
$$y a_y \cdot z b_z = yz a_y b_z$$
$$(\sum_{y \in G} y a_y)(\sum_{z \in G} z b_z) = \sum_{x \in G} x(\sum_{yz = x} a_y b_z).$$

Example

If $G = \mathbb{Z}$, then $\Bbbk[G] = \Bbbk[t, t^{-1}]$.

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Example

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Example of ring with involution

Let $\mathbb{k}$ be a field

Example

If G is a group and $\mathbb{k}[G]$ denotes the group algebra of G over $\mathbb{k}$

$$\begin{array}{ccc} \mathbb{k}[G] & \longrightarrow & \mathbb{k}[G] \\ \sum_{x\in G} x a_x & \longmapsto & \sum_{x \in G} x^{-1} a_x, \end{array}$$

is a $\mathbb{k}$-involution called the canonical involution of $\mathbb{k}[G]$.

Can we obtain a division ring with involution from this?

General situation

If R is a commutative ring:

- **Existence**: A division ring of fractions exists iff R is a domain.
- **Uniqueness**: Division rings of fractions are isomorphic

- **Form of the elements:** Elements of D are fractions $\frac{r}{s} = s^{-1}r$

In general:

- Domains not embeddable in division rings.
- Domains with more than one division ring of fractions.
- Expressions like $r - s(t - uv^{-1}w)^{-1}x$ may not be simplified.

- If we want $\Bbbk[G]$ to be a domain, G has to be torsion free:
  If $x^n = 1$, then $(1-x)(1+x+\cdots+x^{n-1}) = 0$.

- Open problem: when is $\Bbbk[G]$ embeddable in a division ring?

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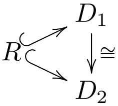

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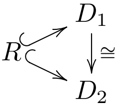

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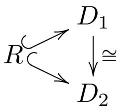

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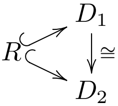

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Malcev-Neumann series ring

Example

$$\Bbbk\mathbb{Z} = k[t, t^{-1}] \hookrightarrow \Bbbk(t) \hookrightarrow \Bbbk((t)) = \Big\{ \sum_{i \geq n} t^i a_i \mid a_i \in \Bbbk,\ n \in \mathbb{Z} \Big\}.$$

Definition

- $(G, <)$ is an ordered group if G is a group and $<$ is a total order such that for all $x, y, z \in G$

$$x < y \Rightarrow xz < yz \qquad x < y \Rightarrow zx < zy$$

- $(G, <)$ ordered group. $\Bbbk$ a field, $\Bbbk[G]$ the group algebra.

$$\Bbbk[G] \hookrightarrow \Bbbk((G, <)) = \Big\{ f = \sum_{x \in G} x a_x \mid a_x \in \Bbbk,\ \operatorname{supp} f \text{ is well ordered} \Big\}$$

  $\Bbbk((G, <))$ is a division ring, Malcev-Neumann series ring.

- $\Bbbk(G)$ is the division ring generated by $\Bbbk[G]$ inside $\Bbbk((G, <))$.

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Some results on group rings

Let $\mathbb{k}$ be a field.

- If G is an orderable group, then $\mathbb{k}[G]$ is embeddable in the division ring $\mathbb{k}((G))$. **[Malcev-Neumann]**
- Any division ring that contains $\mathbb{k}[G]$ must contain a free algebra $\mathbb{k}\langle X\rangle$. **[S.]**

Theorem (Ferreira-Gonçalves-S.)

Let G be an orderable group and $\mathbb{k}[G]$ be the group algebra.

Let $\mathbb{k}(G)$ be the division ring generated by $\mathbb{k}[G]$ inside $\mathbb{k}((G))$.

Then the canonical involution extends to $\mathbb{k}(G)$ and the following are equivalent:

- *$\mathbb{k}(G)$ contains a free $\mathbb{k}$-algebra freely generated by symmetric elements with respect to the canonical involution.*
- *G is not abelian.*

Universal enveloping algebra

Example

Let L be a Lie $\Bbbk$-algebra, and $U(L)$ its universal enveloping algebra.

$$\begin{array}{rcl} U(L) & \longrightarrow & U(L) \\ x & \longmapsto & -x, \qquad \text{for all } x \in L \end{array}$$

is a $\Bbbk$-involution called the principal involution of $U(L)$.

- $U(L)$ embeds in a division ring $\mathfrak{D}(L)$ **(P. M. Cohn)**
- A more manageable construction of $\mathfrak{D}(L)$ **(A. I. Lichtman)**
- If $L_1 \leq L$, then $\mathfrak{D}(L_1) \subseteq \mathfrak{D}(L)$
- If $U(L)$ is an Ore domain, then $\mathfrak{D}(L)$ coincides with its Ore skew field of fractions
- The principal involution extends to a $\Bbbk$-involution $\mathfrak{D}(L) \to \mathfrak{D}(L)$ **(J. Cimprič)**
- If L is not abelian and char $\Bbbk$=0, $\mathfrak{D}(L)$ contains free algebras $\Bbbk\langle X\rangle$ **(A. I. Lichtman)**

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General strategy

- Let D be $\mathbb{k}$-algebra that contains a free $\mathbb{k}$-algebra $\mathbb{k}\langle x, y\rangle$.
- Suppose there exists a morphism of $\mathbb{k}$-algebras $\varphi\colon R \to D$ such that $\varphi(a) = x$ and $\varphi(b) = y$.
- Then the $\mathbb{k}$-algebra generated by $\{a, b\}$ is the free $\mathbb{k}$-algebra $\mathbb{k}\langle a, b\rangle$.
- Problem: Any morphism of rings between division rings is injective and thus an embedding.
- Solution: Find a suitable subring T such that there exists $\varphi\colon T \to D$.

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Main result

From now on, $\mathbb{k}$ is a field of characteristic zero.

Theorem (Ferreira-Gonçalves-S.)

Let L be a nonabelian Lie $\mathbb{k}$-algebra such that either L is residually nilpotent or $U(L)$ is an Ore domain.
Then $\mathfrak{D}(L)$ contains a free algebra $\mathbb{k}\langle X\rangle$ generated by symmetric elements with respect to the principal involution on $\mathfrak{D}(L)$.
Moreover, in these cases, we give explicit symmetric elements that generate the free $\mathbb{k}$-algebra.

Structure of the proof:

- Prove the existence of free algebras generated by symmetric elements for the Lie $\mathbb{k}$-algebra

$$H = \langle x, y \mid [x,[y,x]] = [y,[y,x]] = 0\rangle.$$

- Prove the result for residually nilpotent Lie $\mathbb{k}$-algebra.
- Prove the result when $U(L)$ is an Ore domain.

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- Prove the result when $U(L)$ is an Ore domain.

Heisenberg Lie algebra

L. Makar-Limanov, G. Cauchon

- $\mathbb{k}$ a field of characteristic zero.
- $\sigma\colon \mathbb{k}(t) \to \mathbb{k}(t)$ automorphism determined by $\sigma(t) = t - 1$.
- $\mathbb{k}(t)[p;\sigma] = \left\{\sum_{i=0}^{n} p^i a_i \mid a_i \in \mathbb{k}(t)\right\}$, where

$$ap = p\sigma(a), \text{ for all } a \in \mathbb{k}(t).$$

- $\mathbb{k}(t)(p;\sigma)$ Ore classical ring of quotients of $\mathbb{k}(t)[p;\sigma]$.
- Define

$$s = (t - \frac{5}{6})(t - \frac{1}{6})^{-1}, \quad u = (1 - p^2)(1 + p^2)^{-1}.$$

Then the $\mathbb{k}$-algebra generated by the elements $s + s^{-1}$ and $u(s + s^{-1})u^{-1}$ is a free $\mathbb{k}$-algebra inside $\mathbb{k}(t)(p;\sigma)$.

Heisenberg Lie algebra H

- $H = \langle x, y \mid [x,[y,x]] = [y,[y,x]] = 0\rangle$, define $z = [y,x]$.
- $$\begin{array}{rcl} \Upsilon : \ U(H) & \longrightarrow & \mathbb{k}(t)(p;\sigma) \\ x & \mapsto & p^{-1}t \\ y & \mapsto & p \\ z & \mapsto & 1. \end{array}$$
- $\mathfrak{S} = U(H) \setminus \ker \Upsilon$ is an Ore set of $U(H)$.
- Υ can be extended to a surjective morphism of $\mathbb{k}$-algebras $\Upsilon \colon \mathfrak{S}^{-1}U(H) \to \mathbb{k}(t)(p;\sigma)$.

Heisenberg Lie algebra H

Proposition (Ferreira-Gonçalves-S.)

Consider the Heisenberg Lie $\Bbbk$-algebra

$$H = \langle x, y \mid [x,[y,x]] = [y,[y,x]] = 0 \rangle.$$

Let $U(H)$ be its universal enveloping algebra, and let $\mathfrak{D}(H)$ be its classical Ore division ring of fractions.
Define

$$z = [y,x], \quad V = \frac{1}{2} z(xy + yx)z,$$

$$S = \left(V - \frac{1}{3}z^3\right)\left(V + \frac{1}{3}z^3\right)^{-1} + \left(V - \frac{1}{3}z^3\right)^{-1}\left(V + \frac{1}{3}z^3\right),$$

$$T = (z + y^2)^{-1}(z - y^2)S(z + y^2)(z - y^2)^{-1}.$$

Then:

- *The elements S and T are symmetric with respect to the principal involution on $\mathfrak{D}(H)$, and they generate a free $\Bbbk$-algebra of rank 2.*

Residually nilpotent Lie algebra

- R a ring with $\delta\colon R \to R$ a derivation.
- $R[x;\delta] = \{\sum_{i=0}^{n} x^i a_i \mid a_i \in R\}$, where

$$ax = xa + \delta(a), \quad \text{for all } a \in R.$$

- Define $t_x = x^{-1}$, then

$$R[x;\delta] \hookrightarrow R((t_x;\delta)) \ = \ \Big\{ \sum_{i \geq N} t_x^i a_i \mid a_i \in R \Big\}.$$

Residually nilpotent Lie algebra

- $H = \langle x, y \mid [x,[y,x]] = [y,[y,x]] = 0\rangle,\ \ z = [y,x]$.
- $U(H) = \Bbbk[z][y][x;\delta_x] \hookrightarrow \Bbbk((t_z))((t_y))((t_x;\delta_x))$.
- Let L be a Lie $\Bbbk$-algebra generated by $\{u, v\}$. Define $w = [v, u]$. Suppose that there exists a morphism of Lie algebras

$$L \to H, \quad u \mapsto x,\ v \mapsto y.$$

  Let N be the kernel. Thus $L/N \cong N$. Then

$$U(L) = U(N)[w;\delta_w][v;\delta_v][u;\delta_u] \hookrightarrow U(N)((t_w;\delta_w))((t_v;\delta_v))((t_u;\delta_u)).$$

- Augmentation map $\varepsilon\colon U(N) \to \Bbbk,\ \ n \mapsto 0$ for all $n \in N$.

Residually nilpotent Lie algebra

- $H = \langle x, y \mid [x,[y,x]] = [y,[y,x]] = 0\rangle,\ z = [y,x]$.
- $U(H) = \mathbb{k}[z][y][x;\delta_x] \hookrightarrow \mathbb{k}((t_z))((t_y))((t_x;\delta_x))$.
- Let L be a Lie $\mathbb{k}$-algebra generated by $\{u, v\}$. Define $w = [v, u]$. Suppose that there exists a morphism of Lie algebras

  $$L \to H, \quad u \mapsto x,\ v \mapsto y.$$

  Let N be the kernel. Thus $L/N \cong N$. Then

  $$U(L) = U(N)[w;\delta_w][v;\delta_v][u;\delta_u] \hookrightarrow U(N)((t_w;\delta_w))((t_v;\delta_v))((t_u;\delta_u)).$$

- Augmentation map $\varepsilon\colon U(N) \to \mathbb{k},\ n \mapsto 0$ for all $n \in N$.

Residually nilpotent Lie algebra

$$
\begin{array}{ccc}
U(L) = U(N)[w;\delta_w][v;\delta_v][u;\delta_u] & \longrightarrow & \Bbbk[z][y][x;\delta_x] = U(H) \\
\Big\downarrow & & \Big\downarrow \\
 & & \mathfrak{D}(H) \\
 & & \Big\downarrow \\
U(N)((t_w;\delta_w))((t_v;\delta_v))((t_u;\delta_u)) & \longrightarrow & \Bbbk((t_z))((t_y))((t_x;\delta_x))
\end{array}
$$

$$
\begin{array}{ccccc}
U(N)[w;\delta_w][v;\delta_v][u;\delta_u] & & \hookrightarrow & & \mathfrak{D}(N)[w;\delta_w][v;\delta_v][u;\delta_u] \\
\Big\downarrow & \searrow & & \swarrow & \Big\downarrow \\
 & & \mathfrak{D}(L) & & \\
 & & & \searrow & \\
U(N)((t_w;\delta_w))((t_v;\delta_v))((t_u;\delta_u)) & & \hookrightarrow & & \mathfrak{D}(N)((t_w;\delta_w))((t_v;\delta_v))((t_u;\delta_u))
\end{array}
$$

Residually nilpotent Lie algebra

$$\begin{array}{ccc}
U(L) = U(N)[w;\delta_w][v;\delta_v][u;\delta_u] & \longrightarrow & \Bbbk[z][y][x;\delta_x] = U(H) \\
\big\downarrow & & \big\downarrow \\
 & & \mathfrak{D}(H) \\
 & & \big\downarrow \\
U(N)((t_w;\delta_w))((t_v;\delta_v))((t_u;\delta_u)) & \longrightarrow & \Bbbk((t_z))((t_y))((t_x;\delta_x))
\end{array}$$

$$\begin{array}{ccccc}
U(N)[w;\delta_w][v;\delta_v][u;\delta_u] & & \hookrightarrow & & \mathfrak{D}(N)[w;\delta_w][v;\delta_v][u;\delta_u] \\
\big\downarrow & \searrow & & \swarrow & \big\downarrow \\
 & & \mathfrak{D}(L) & & \\
 & & & \searrow & \\
U(N)((t_w;\delta_w))((t_v;\delta_v))((t_u;\delta_u)) & & \hookrightarrow & & \mathfrak{D}(N)((t_w;\delta_w))((t_v;\delta_v))((t_u;\delta_u))
\end{array}$$

Residually nilpotent Lie algebra

Theorem (Ferreira-Gonçalves-S.)

Let $H = \langle x, y \mid [[y,x],x] = [[y,x],y] = 0 \rangle$ *be the Heisenberg Lie* $\mathbb{k}$*-algebra and let* L *be a Lie* $\mathbb{k}$*-algebra generated by two elements* u, v*. Suppose that there exists a Lie* $\mathbb{k}$*-algebra homomorphism*

$$L \to H, \ u \mapsto x, \ v \mapsto y. \tag{1}$$

Let $w = [v,u]$*,* $V = \frac{1}{2}w(uv+vu)w$*, and consider the following elements of* $\mathfrak{D}(L)$*:*

$$S = (V - \frac{1}{3}w^3)(V + \frac{1}{3}w^3)^{-1} + (V - \frac{1}{3}w^3)^{-1}(V + \frac{1}{3}w^3),$$

$$T = (w+v^2)^{-1}(w-v^2)S(w+v^2)(w-v^2)^{-1}.$$

Then:

- *The elements* S *and* T *are symmetric with respect to the principal involution on* $\mathfrak{D}(L)$ *and they generate a free* $\mathbb{k}$*-algebra of rank 2.*

In a residually nilpotent Lie $\mathbb{k}$-algebra, the Lie subalgebra L generated by two noncommuting elements u, v satisfies the condition (1).

$U(L)$ is an Ore domain

Theorem (Ferreira-Gonçalves-S.)

Let L be a Lie $\mathbb{k}$-algebra such that its universal enveloping algebra $U(L)$ is an Ore domain, and let $\mathfrak{D}(L)$ be its classical Ore division ring of fractions.
Let $u, v \in L$ such that the Lie subalgebra generated by them is of dimension at least three. Define

$$w = [u, v], \quad V = \frac{1}{2}w(uv + vu)w,$$

$$S = \left(V - \frac{1}{3}w^3\right)\left(V + \frac{1}{3}w^3\right)^{-1} + \left(V - \frac{1}{3}w^3\right)^{-1}\left(V + \frac{1}{3}w^3\right),$$

$$T = (w + u^2)^{-1}(w - u^2)S(w + u^2)(w - u^2)^{-1}.$$

Then:

- *The elements S and T are symmetric with respect to the principal involution on $\mathfrak{D}(L)$ and they generate a free $\mathbb{k}$ -algebra or rank two.*

If the dimension of the Lie subalgebra generated by u and v is of dimension two, use the result by **Cauchon**.

$U(L)$ is an Ore domain

Technique by A. I. Lichtman

- Can suppose that L is generated by two elements u, v.
- Obtain a filtration of $U(L)$:

$$\Bbbk = U_0(L) \subseteq U_{-1}(L) \subseteq \cdots \subseteq U_{-n}(L) \subseteq \cdots$$

$$U_{-n}(L) = \Bbbk - \text{subspace gen. products of } \leq n \text{ elements from } \{u, v\}.$$

- Obtain a filtration of L:

$$L_{-n} = L \cap U_{-n}(L).$$

- $\mathrm{gr}(U(L)) \cong U(\mathrm{gr}(L)) \implies \mathrm{gr}(U(L))$ a domain
- Filtration induces a valuation $\vartheta \colon U(L) \to \mathbb{Z} \cup \{\infty\}$
- Can be extended to a valuation $\vartheta \colon U(L)[t, t^{-1}] \to \mathbb{Z} \cup \{\infty\}$
- $T = \{f \in U(L)[t, t^{-1}] \mid \vartheta(f) \geq 0\}$,
  $T_0 = \{f \in U(L)[t, t^{-1}] \mid \vartheta(f) > 0\}$

$U(L)$ is an Ore domain

- Valuation $\vartheta\colon U(L)[t,t^{-1}] \to \mathbb{Z} \cup \{\infty\}$
- $T = \{f \in U(L)[t,t^{-1}] \mid \vartheta(f) \geq 0\}$,
  $T_0 = \{f \in U(L)[t,t^{-1}] \mid \vartheta(f) > 0\}$
- $$\begin{array}{rcl} T/T_0 & \xrightarrow{\cong} & U(\mathrm{gr}(L)) \cong \mathrm{gr}(U(L)) \\ ut + T_0 & \mapsto & \bar{u} \\ vt + T_0 & \mapsto & \bar{v} \\ wt^2 + T_0 & \mapsto & \bar{w} \end{array}$$
- $U(L)$ Ore domain $\Rightarrow T$, $U(\mathrm{gr}(L))$ are Ore domains.
- Let $\mathfrak{D}(L)(t)$ and Δ be the Ore ring of fractions of T and $U(\mathrm{gr}(L))$ respect.
- $\mathrm{gr}(L)$ is a non commutative residually nilpotent Lie $\Bbbk$-algebra
- Let $\mathfrak{S} = T \setminus T_0$ is an Ore subset of T.
- $\mathfrak{S}^{-1}T \longrightarrow U(\mathrm{gr}(L))$, $ut \mapsto \bar{u}$, $vt \mapsto \bar{v}$, $wt^2 \mapsto \bar{w}$.
- When we perform the operations to obtain the free algebra in $\mathfrak{S}^{-1}T \subseteq \mathfrak{D}(L)(t)$, amazingly enough the elements are in $\mathfrak{D}(L)$.

Muchas gracias