Aportes de ITENU a la Resolución de Problemas de Números de Fibonacci

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This group was created during the fourth international conference on Algebra, Number Theory, Combinatorics and Applications in Tunja, Colombia, ALTENCOA4-2010.

We are focused on the number theory, we are dedicated specifically to solve problems The Fibonacci Quarterly.

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Lines of research

- Fibonacci numbers.
- Theory of partitions.
- Theory of compositions.
- Theory of representations.
- Figurate numbers.

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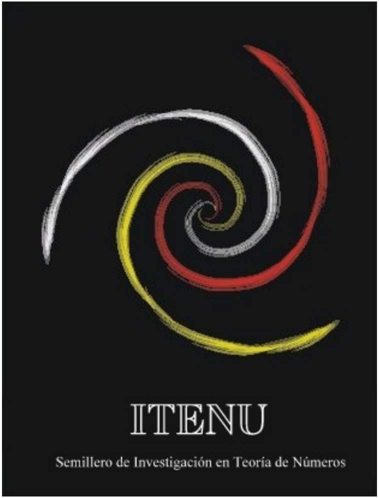

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Fibonacci Quarterly

The Fibonacci Quarterly is a scientific journal on mathematical topics related to the Fibonacci numbers, published four times by year. It is the primary publication of The Fibonacci Association, which has published it since 1963.

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B-1142

Theorem

Prove that

$$\sum_{k=1}^{n} F_{4k-1} = F_{2n}F_{2n+1} \text{ for any positive integer } n.$$

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Proof.

We know that $F_{2n+1} = F_n^2 + F_{n+1}^2$ and $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, see for example [**?**, p.79, Cor. 5.4] and [**?**, p.77, Th. 5.5], respectively. Thus,

$$\begin{aligned}\sum_{k=1}^{n} F_{4k-1} &= \sum_{k=1}^{n} \left(F_{2k}^2 + F_{2k-1}^2\right)\\ &= \sum_{k=1}^{2n} F_k^2\\ &= F_{2n} F_{2n+1}.\end{aligned}$$

This proves the result. □

B-1146

Theorem

Prove that

$$F_{n+2}^2 \geq 5F_{n-1}^2 \text{ for all integer } n \geq 1.$$

B-1146

Proof.

We prove the inequality by strong induction on n: The inequality is clear when $n = 1$ and when $n = 2$, We now assume that the inequality is true to a natural number n. Since $F_{n+2}^2 \geq 5F_{n-1}^2$, $F_{n+1}^2 \geq 5F_{n-2}^2$ and $2F_{n+1}F_{n+2} \geq 2(5F_{n-2}F_{n-1})$, we have that:

$$
\begin{aligned}
F_{n+3}^2 &= (F_{n+2} + F_{n+1})^2 \\
&= F_{n+2}^2 + 2F_{n+2}F_{n+1} + F_{n+1}^2 \\
&\geq 5F_{n-1} + 2(5F_{n-2}F_{n-1}) + 5F_{n-2} \\
&= 5(F_{n-1} + F_{n-2})^2 \\
&= 5F_n^2
\end{aligned}
$$

This proves the result. $\square$

B-1144

Theorem

Prove that

$$\prod_{k=1}^{n}(F_k^2+1) \geq F_nF_{n+1}+1$$
$$\prod_{k=1}^{n}(L_k^2+1) \geq L_nL_{n+1}-1$$

For any positive integer n.

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$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}, \quad [1, p.77, Id.5.5]$$

$$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2, \quad [1, p.78, Id.5.10]$$

B-1144

Lemma

Let $\{G_n\}_{n\in\mathbb{N}}$ *be the generalized Fibonacci sequence with integers* a *and* b. *That is,* $G_1 = a$, $G_2 = b$ *and* $G_n = G_{n-1} + G_{n-2}$ *for all* $n \in \mathbb{N}\backslash\{1, 2\}$, *then*

$$\sum_{k=1}^{n} G_k^2 = G_n G_{n+1} + a(a - b)$$

B-1144

Proof.

(by PMI) When $n = 1$:

$\sum_{k=1}^{1} G_k^2 = G_1^2 = a^2 = a^2 - ab + ab = G_1 G_2 + a(a-b)$. So the result is true when $n = 1$.

Assume it is true for an arbitrary positive integer t:

$$\sum_{k=1}^{t} G_k^2 = G_t G_{t+1} + a(a-b).$$

B-1144

Proof.

Then

$$\begin{aligned}\sum_{k=1}^{t+1} G_k^2 &= \left(\sum_{k=1}^{t} G_k^2\right) + G_{t+1}^2 \\ &= G_t G_{t+1} + a(a-b) + G_{t+1}^2 \qquad \textit{by the IH} \\ &= G_{t+1}(G_t + G_{t+1}) + a(a-b) \\ &= G_{t+1} G_{t+2} + a(a-b)\end{aligned}$$

So the statement is true when $n = t + 1$. Thus it is true for every positive integer n. $\square$

B-1144

Lemma

If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers, then

$$\prod_{k=1}^{n}(x_k+1) \geq \left(\sum_{k=1}^{n} x_k\right) + 1.$$

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Proof.

(by PMI) When $n = 1$, $\prod_{k=1}^{1}(x_k + 1) = x_1 + 1 = \left(\sum_{k=1}^{1} x_k\right) + 1$.
So the result is true when $n = 1$.

Assume it is true for an arbitrary positive integer t:

$$\prod_{k=1}^{t}(x_k + 1) \geq \left(\sum_{k=1}^{t} x_k\right) + 1.$$

B-1144

Proof.

Then

$$\begin{aligned}
\prod_{k=1}^{t+1}(x_k+1) &= \left(\prod_{k=1}^{t}(x_k+1)\right)(x_{t+1}+1) \\
&\geq \left[\left(\sum_{k=1}^{t}x_k\right)+1\right](x_{t+1}+1) \qquad \textit{by the IH} \\
&\geq x_{t+1}\left(\sum_{k=1}^{t}x_k\right)+\left(\sum_{k=1}^{t+1}x_k\right)+1 \\
&\geq \left(\sum_{k=1}^{t+1}x_k\right)+1
\end{aligned}$$

$\square$

B-1144

Theorem

Prove that

$$\prod_{k=1}^{n}(F_k^2+1) \geq F_nF_{n+1}+1$$

$$\prod_{k=1}^{n}(L_k^2+1) \geq L_nL_{n+1}-1$$

For any positive integer n.

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Proof.

Let $\{G_n\}_{n\in\mathbb{N}}$ be the generalized Fibonacci sequence with integers a and b. That is, $G_1 = a$, $G_2 = b$ and $G_n = G_{n-1} + G_{n-2}$ for all $n \in \mathbb{N}\backslash\{1, 2\}$. Note that if $a = b = 1$, then $G_n = F_n$. If $a = 1$ and $b = 3$, then $G_n = L_n$.

This way, can be proved a more general result. I.e.:

$$\prod_{k=1}^{n}(G_k^2 + 1) \geq G_nG_{n+1} + a(a - b) + 1 \quad \textit{for all } n \in \mathbb{N}$$

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Proof.

So

$$\prod_{k=1}^{n}(G_k^2+1) \geq \left(\sum_{k=1}^{n} G_k^2\right) + 1$$

$$\prod_{k=1}^{n}(G_k^2+1) \geq G_k G_{k+1} + a(a-b) + 1$$

$\square$

Bibliography

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