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Hamiltonian Formulation of the Yang–Mills field on the null–plane

R. Casana[a] *, B.M. Pimentel[b†] and G. E. R. Zambrano[b‡ §]

[a]Universidade Federal do Maranhão (UFMA), Departamento de Física, Campus Universitário do Bacanga, CEP 65085-580, São Luís - MA, Brasil.

[b]Instituto de Física Teórica (IFT/UNESP), UNESP - São Paulo State University, Caixa Postal 70532-2, 01156-970, São Paulo, SP, Brazil

We have studied the null–plane hamiltonian structure of the free Yang–Mills fields. Following the Dirac's procedure for constrained systems we have performed a detailed analysis of the constraint structure of the model and we give the generalized Dirac brackets for the physical variables. Using the correspondence principle in the Dirac's brackets we obtain the same commutators present in the literature and new ones.

1. Introduction

To quantize the theory on the null–plane [1], initial conditions on the hyperplane $x^+ = cte$ and equal x^+-commutation relations must be given and the hamiltonian must describe the time evolution from an initial value surface to other parallel surface that intersects the x^+-axis at some later time. Inside the null–plane framework, the lagrangian which describes a given field theory is singular, thus, the Dirac's method [2] allows to build the null–plane hamiltonian and the canonical commutation relations in terms of the independent fields of the theory.

It is interesting to observe that the null–plane quantization of a non-abelian gauge theory using the null–plane gauge condition, $A_- = 0$, identified the transverse components of the gauge field as the degrees of freedom of the theory and, therefore, the ghost fields can be eliminated of the quantum action [3].

Tomboulis has quantized the massless Yang–Mills field in the null–plane gauge $A^a_- = 0$ and has derived the Feynman rules [4]. However, it was shown that the null–plane quantization of this theory leads a set of second–class constraints in addition to the usual first–class constraints, characteristics of the usual instant form quantization, which leads to the introduction of additional ghost fields in the effective lagrangian [5]. Moreover, the theory has been quantized in the framework of the standard perturbation approach and it was explained that the difficulties appearing in the null–plane gauge are overcome using the gauge $A^a_+ = 0$, such gauge provides a generating functional for the renormalized Green's functions that takes to the Mandelstam–Leibbrandt's prescription for the free gluon propagator [6].

In this paper we will discuss the null–plane structure of the pure Yang–Mills fields following Dirac's formalism for constrained systems. The work is organized as follows: In the section 2, we study the free Yang–Mills field, its constrained structure being analysed in detail, thus, we classify the constraints of the theory. In the section 3 the appropriated equations of motion of the dynamical variables are determined by using the extended hamiltonian, and the null–plane gauge is imposed to transform the set of first class constraints into second–class ones. In the section 4 the Dirac's brackets (DB) among the independent fields are obtained by choosing appropriate boundary conditions on the fields. Finally, we give our conclusions and remarks.

*casana@ufma.br
†pimentel@ift.unesp.br
‡gramos@ift.unesp.br
§On leave of absence from Departamento de Física, Universidad de Nariño, San Juan de Pasto, Nariño, Colombia.

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2. Free Yang–Mills field

For any semi-simple Lie group with structure constant f^a_{bc} the Yang-Mill lagrangian density is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu}, \tag{1}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{bc} A^b_\mu A^c_\nu$, the gauge index a, b, c runs from 1 to n. Such lagrangian is invariant under the following infinitesimal gauge transformations

$$\delta A^\mu_a(x) = f^a_{bc} \Lambda^b(x) A^\mu_c(x) + \frac{1}{g} \partial^\mu \Lambda_a(x). \tag{2}$$

with $\Lambda_a(x)$ an arbitrary function.

In the present work, we specialize for convenience to the $SU(2)$ gauge group that only has three generators and $f^a_{bc} = \varepsilon_{abc}$, where ε_{abc} is the Levi-Civita totally antisymmetric tensor in three dimensions, thus, we can define everything in such way that we can forget about raising and lowering group indices. From (1) we find the Euler–Lagrange equations

$$(D_\nu)^{ab} F^{\nu\mu}_b = 0, \tag{3}$$

where we have defined the covariant derivative defined as

$$(D_\nu)^{ab} \equiv \delta^a_b \partial_\nu - g \varepsilon_{abc} A^c_\nu.$$

2.1. Structure Constraints and Classification

In the null–plane dynamics, the canonical conjugate momenta are

$$\pi^\mu_a \equiv \frac{\partial \mathcal{L}}{\partial(\partial_+ A^a_\mu)} = -F^{+\mu}_a , \tag{4}$$

this equation gives the following set of primary constraints

$$\begin{aligned} \phi_a &\equiv \pi^+_a \approx 0, \\ \phi^k_a &\equiv \pi^k_a - \partial_- A^a_k + \partial_k A^a_- - g\varepsilon_{abc} A^b_- A^c_k \approx 0 . \end{aligned} \tag{5}$$

and the dynamical relation for A^a_-

$$\pi^-_a = \partial_+ A^a_- - \partial_- A^a_+ - g \varepsilon_{abc} A^b_+ A^c_- , \tag{6}$$

Immediately, the canonical hamiltonian is given by

$$\begin{aligned} H_C = \int d^3y \Big\{ &\frac{1}{2}\left(\pi^-_a\right)^2 + \pi^-_a (D_-)^{ab} A^b_+ \\ &+ \pi^i_a (D_i)^{ab} A^b_+ + \frac{1}{4}\left(F^a_{ij}\right)^2 \Big\}. \end{aligned} \tag{7}$$

Following the Dirac procedure [2], we define the primary hamiltonian adding to the canonical hamiltonian the primary constraints

$$\begin{aligned} H_P = \int d^3y \Big\{ &\frac{1}{2}\left(\pi^-_a\right)^2 + \pi^-_a (D_-)^{ab} A^b_+ \\ &+ \pi^i_a (D_i)^{ab} A^b_+ + \frac{1}{4}\left(F^a_{ij}\right)^2 + u^b \phi_b + \lambda^b_l \phi^l_b \Big\} \end{aligned} \tag{8}$$

where u^b and λ^b_l are their respective Lagrange multipliers.

The fundamental Poisson brackets (PB) among fields are

$$\left\{A^a_\mu(x), \pi^\nu_b(y)\right\} = \delta^\nu_\mu \delta^a_b \delta^3(x-y). \tag{9}$$

Requiring that H_P is the generator of temporal evolutions, the consistency condition of the primary constraints, i.e. $\{\phi, H_P\} = 0$, give us for ϕ_a

$$\dot{\phi}_a = (D_-)^{ab} \pi^-_b + (D_i)^{ab} \pi^i_b \equiv G_a \approx 0 , \tag{10}$$

a genuine secondary constraint, which is the Gauss's law. Also, for ϕ^k_a we obtain

$$\begin{aligned} \dot{\phi}^k_a = \; &(D_k)^{ab} F^b_{+-} + (D_i)^{ab} F^b_{ik} \\ &-2 (D_-)^{ab} \lambda^b_k \approx 0 , \end{aligned} \tag{11}$$

a differential equation which allows to compute λ^b_k after imposition of appropriated boundary conditions. The consistency condition of the secondary constraint yields

$$\{G_a(x), H_P\} = g \varepsilon_{acb} A^c_+(x) G_b(x) \approx 0, \tag{12}$$

thus, the Gauss's law is automatically conserved. Then, there are no more constraints and the equations (5) and (10) give the full set of constraints.

The set of first class constraints is $\{\pi^+_a , G_a\}$ and the set of second class constraints is $\left\{\phi^k_a\right\}$ whose PB's are

$$\left\{\phi^k_a(x), \phi^l_b(y)\right\} = -2\delta^l_k (D_-)^{ab} \delta^3(x-y), \tag{13}$$

we observe that of here and from now the derived operator acts on the x-coordinate.

3. Equation of motion

Now we check the equations of motion. The time evolution of the fields is determined by computing their PB's with the so called extended hamiltonian H_E, which is obtained by adding to the primary hamiltonian all the first class constraints of the theory:

$$\mathrm{H}_E = \mathrm{H}_C + \int d^3y \left\{ \lambda_l^b \phi_b^l + \mathrm{u}^b \phi_b + \mathrm{v}^b G_b \right\} \tag{14}$$

thus, we have the time evolution of the dynamical variables, i.e, $\dot{\phi} = \{\phi, H_E\}$, gives

$$\dot{A}_+^a = \mathrm{u}^a \tag{15}$$

$$\dot{A}_-^a = \pi_a^- + (D_-)^{ac} A_+^c - (D_-)^{ab} \mathrm{v}^b \tag{16}$$

$$\dot{A}_k^a = (D_k)^{ac} A_+^c + \lambda_k^a - (D_k)^{ab} \mathrm{v}^b \tag{17}$$

$$\dot{\pi}_a^+ = G_a \tag{18}$$

$$\dot{\pi}_a^- = -g\varepsilon_{abc}\pi_b^- A_+^c + (D_l)^{ab} \lambda_l^b - g\varepsilon_{bca}\mathrm{v}^b\pi_c^- \tag{19}$$

$$\dot{\pi}_a^k = -g\varepsilon_{bca}\pi_b^k A_+^c + (D_j)^{ab} F_{kj}^b - (D_-)^{ab} \lambda_k^b - g\varepsilon_{abc}\pi_c^k \mathrm{v}^b \ . \tag{20}$$

If we demand consistency with the Euler–Lagrange equations of motion (3) we must choose $\mathrm{v}^b = 0$, however, the multiplier u^a remains indeterminate.

Dirac's algorithm requires as many gauge conditions as there are first–class constraints, nevertheless these conditions should be compatible with the Euler–Lagrange equations and together with the first class set they should form a second class set, in such way that the Lagrange multipliers, corresponding to the first class set, are determined. Under such considerations, we choose as the first gauge condition

$$A_-^a \approx 0, \tag{21}$$

whose consistency condition $\dot{A}_-^a = \left\{A_-^a, H_E\right\} \approx 0$ must be compatible with the dynamical equation (6) thus we see that if we choose $\mathrm{v}^b = 0$ in (16) then the Eq.(21) will hold for all times only if

$$\pi_a^- + \partial_-^x A_+^a \approx 0 \ . \tag{22}$$

Therefore, equations (21) and (22) constitute our gauge conditions on the null–plane and they are known as the null–plane gauge.

4. Dirac Brackets

The prescription for determining the Dirac brackets implies calculating the inverse of the second–class matrix. We rename the second–class constraints as follows

$$\begin{aligned}
\Theta_1 &\equiv \pi_a^+ \quad , \quad \Theta_2 \equiv (D_-)^{ab} \pi_b^- + (D_i)^{ab} \pi_b^i \\
\Theta_3 &\equiv A_-^a \quad , \quad \Theta_4 \equiv \pi_a^- + \partial_- A_+^a \\
\Theta_5 &\equiv \pi_a^k - \partial_- A_k^a + \partial_k A_-^a - g\varepsilon_{abc} A_-^b A_k^c,
\end{aligned} \tag{23}$$

and we define the elements of the second class matrix as $F_{ab}\left(x,y\right) \equiv \left\{\Theta_a\left(x\right), \Theta_b\left(y\right)\right\}$. The Dirac's brackets between two dynamical variables of the theory is determined if the inverse of the second class constraint matrix is calculated explicitly. Now, the evaluation of F^{-1} involves the determination of an arbitrary function of the variables x^+ and $x^\perp$ [7] which can be fixed by considering appropriate boundary conditions [8] on the fields A_μ^a Thus, we obtain the DB among the independent variables of the theory

$$\left\{A_k^a(x), A_l^b(y)\right\}_D = -\frac{1}{4}\delta_b^a\delta_k^l\epsilon(x-y)\delta^2\left(x^\perp - y^\perp\right) \tag{24}$$

$$\left\{A_k^a(x), A_+^b(y)\right\}_D = \frac{1}{4}|x-y|\left(D_k\right)^{ab}\delta^2\left(x^\perp - y^\perp\right).$$

Immediately, via the correspondence principle we obtain the commutators among the fields

$$\left[A_k^a(x), A_l^b(y)\right] = -\frac{i}{4}\delta_b^a\delta_k^l\epsilon(x-y)\,\delta^2\left(x^\perp - y^\perp\right), \tag{25}$$

$$\left[A_k^a(x), A_+^b(y)\right] = \frac{i}{4}\,|x-y|\left(D_k\right)^{ab}\delta^2\left(x^\perp - y^\perp\right). \tag{26}$$

The first relationship is exactly that obtained by Tomboulis [4], but Eq. (26) is a new commutation relation.

5. Remarks and conclusions

In this work we have studied the null–plane Hamiltonian structure of the free Yang–Mills field. Performing a careful analysis of the constraint structure of Yang–Mills field, we have determined in addition to the usual set of first–class constraints, a second–class one, which is a characteristic of the null–plane dynamics [7]. The imposition of appropriated boundary conditions on the fields fixes the hidden subset of first class constraints [9] and eliminates the ambiguity on the operator ∂_-, that allows to get a unique inverse for the second class constraint matrix [7]. The Dirac brackets of the theory are quantized via correspondence principle same as derived by Tomboulis [4].

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