$SQED_4$ and QED_4 on the null-plane

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We studied the scalar electrodynamics ($SQED_4$) and the spinor electrodynamics (QED_4) in the null-plane formalism. We followed the Dirac's technique for constrained systems to perform a detailed analysis of the constraint structure in both theories. We imposed the appropriated boundary conditions on the fields to fix the hidden subset first class constraints which generate improper gauge transformations and obtain an unique inverse of the second class constraint matrix. Finally, choosing the null-plane gauge condition, we determined the generalized Dirac brackets of the independent dynamical variables which via the correspondence principle give the (anti)-commutators for posterior quantization.

PACS numbers: 11.15.Bt ; 03.50.-z
Keywords: Null-plane coordinates, Constraint analysis, Null-plane gauge, Dirac brackets.

I. INTRODUCTION

Half the of last century Dirac [1] proposed three different forms of relativistic dynamics depending on the types of surfaces where the independent degree of freedom was initiated. The first one, named the *instant form*, is when a space-like surface is chosen to establish the fundamental Poisson brackets or commutations relations. It has been used most frequently so far and is usually called equal-time quantization. The second form, the *point form*, is to take a branch of hyperbolic surface $x^\mu x_\mu = \kappa^2$ with $x^0 > 0$. And, the third form, *front form or light front*, is when we choose the surface of a single light wave to study the field dynamics; it is commonly referred as the *null-plane formalism* and it took almost 30 years for Dirac's idea was applied in physical phenomena. An important advantage pointed out by Dirac is that seven of the ten Poincaré generators are kinematical on the null-plane while in the conventional theory constructed on the instant form only six have this property. Other notable feature of a relativistic theory on the null-plane is that it gives origin to singular Lagrangians, e.g. constrained dynamical systems, thus, Dirac's procedure [2] can be employed to analyze the constraint structure of a given theory. In general, it leads to a reduction in the number of independent field operators in the respective phase space.

At equal-time, any two different points are space-like separated and therefore the fields defined at these points are naturally independent quantities. In a null-plane surface the situation is different because the micro-causality principle leads to locality requirement in which only the transversal components are and the longitudinal component becomes non-local in the theory, although, such situation would not be unexpected [3]. It is possible to verify that the transformation from the usual coordinates to the null-plane coordinates is not a Lorentz transformation and the structure of the phase space is different when we compare with the conventional one. As such, the description of a physical system in the null-plane formalism could give additional information from those provided by the conventional formalism [3]. For example, the momentum four-vector is (k^+, k^-, k^T) where k^+ is the null-plane energy while k^T and k^- indicates the transverse and the longitudinal components of the momentum. Therefore, a massive particle on the mass shell, $k^- = \frac{m^2+(k^T)^2}{2k^+}$, has positive definite values for $k\pm$ in contrast to $-\infty \leq k^{1,2,3} \leq$ for the usual components. An immediate consequence is that the vacuum on the null-plane quantized theory may become simpler than the one

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in the conventional (equal-time) theory and in many cases the interacting theory vacuum on the null-plane may be the same as the perturbation theory vacuum. For example, the conservation of the total longitudinal momentum would not permit the excitations of particle-antiparticle pairs by the null-plane vacuum (having $k^+ = 0$) [4].

It was observed that the quantization in the null-plane means to perform the quantization on the characteristic surfaces of the classical field equations. Thus, it implies that one has to specify the Cauchy data on both characteristics, $x^+ = cte$ and $x^- = cte$, and not only on one simple null-plane [5]. In this context, the light cone quantization of free massless fermions in $(1+1)$–dimensions on both characteristics shows that the procedure leads to the correct physical descriptions [6].

On the other hand, in [7] it was showed an important problem associated with the quantization on the null-plane: After establishing the gauge fixing condition for the first class constraints and the second class constraints have seen handled through the Dirac's procedure, no more proper gauge transformations, correspondent to first class constraints, can be performed; however, it still remains in the analysis a species of improper gauge transformations associated to the existence of hidden first class constraints and which are related with the zero mode of the longitudinal derivative ∂_- and appears due the lack of appropriate boundary conditions over the fields [8]. This fact does not allow to define a unique inverse for the second class constraint matrix that is used to define consistent Dirac brackets (DB), therefore, the improper gauge transformations must be fixed by imposing appropriate boundary conditions.

The present work is addressed to study the constraints structure of the scalar and the spinor electrodynamics on the null-plane following the Dirac's formalism for constrained systems. The paper is organized as follow: In the section **2** we study the $SQED_4$, its constraints structure is analyzed in detail and the appropriate equations of motion of the dynamical variables is determined using the extended Hamiltonian. We classify the set of constraints and find that one of the first class constraints is a linear combination of scalar and electromagnetic constraints which is a null vector of the respective constraint matrix. We invert the first class constraints with the corresponding gauge conditions and an unique inverse of the matrix of second class constraints is getting by imposing appropriate boundary conditions on the fields which eliminate the hidden first class constraints and next we calculate the Dirac's brackets among the fundamental dynamical variables. In section **3** we study the QED_4, showing that the use of the projection of the fermionic fields allows to observe the existence of only second class constraints and the first class constraints in the fermionic sector are associated with the hidden subset of first class constraints which generate improper gauge transformations. We also show that the fermionic constraints determine that the electron field is fully described by only two of the four components. We use the null-plane gauge to transform the set of first class constraints in second class and we obtain a graded algebra imposing boundary conditions on the independent components. In the last section we give our remarks and conclusions.

II. SCALAR ELECTRODYNAMICS ($SQED_4$): CONSTRAINT STRUCTURE

The gauge theory we are considering is defined by the following Lagrangian density in 4-dimensional space-time

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + g^{\mu\nu}D_\mu\phi\,(D_\nu\phi)^* - m^2\phi\phi^*, \tag{1}$$

here ϕ is a one-component complex scalar field, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu \equiv \partial_\mu + igA_\mu$ is the covariant derivative. The model is invariant under the following local $U(1)$ gauge symmetry

$$\phi \to e^{i\alpha(x)}\phi \quad , \quad \phi^* \to e^{-i\alpha(x)}\phi^* \quad , \quad A_\mu \to A_\mu - \frac{1}{g}\partial_\mu\alpha. \tag{2}$$

The field equations are given for

$$\partial_\mu F^{\mu\alpha} + j^\alpha = 0 \tag{3}$$

and

$$\left(D^*_\mu D^{*\mu} + m^2\right)\phi^* = 0 \qquad , \qquad \left(D_\mu D^\mu + m^2\right)\phi = 0 \tag{4}$$

where j^α is the current defined by

$$j^\mu \equiv ig\left[\phi\left(\partial^\mu\phi^* - igA^\mu\phi^*\right) - \phi^*\left(\partial^\mu\phi + igA^\mu\phi\right)\right]. \tag{5}$$

The canonical conjugate momenta of the fields $A_\mu,\ \phi$ and ϕ^* are

$$\pi^\mu \equiv \frac{\partial\mathcal{L}}{\partial\left(\partial_+A_\mu\right)} = F^{\mu+}, \tag{6}$$

$$\pi^* \equiv \frac{\partial \mathcal{L}}{\partial \left(\partial_+ \phi\right)} = \left(D_- \phi\right)^* \qquad , \qquad \pi \equiv \frac{\partial \mathcal{L}}{\partial \left(\partial_+ \phi^*\right)} = D_- \phi \tag{7}$$

respectively. Then, from (6) and (7) we get one dynamical relation

$$\pi^- = \partial_+ A_- - \partial_- A_+ \tag{8}$$

and five primary constraints, three for the electromagnetic sector

$$C \equiv \pi^+ \approx 0 \qquad , \qquad \chi^k \equiv \pi^k - \partial_- A_k + \partial_k A_- \approx 0 \tag{9}$$

and, two for the scalar sector

$$\Gamma \equiv \pi - D_- \phi \approx 0 \qquad , \qquad \Gamma^* \equiv \pi^* - \left(D_- \phi\right)^* \approx 0. \tag{10}$$

Following the Dirac's procedure [2], we define the canonical Hamiltonian density which is given by

$$\mathcal{H}_C = \frac{1}{2}\left(\pi^-\right)^2 + \left(\pi^- \partial_- + \pi^k \partial_k - j^+\right) A_+ - D_k \phi \left(D^k \phi\right)^* + m^2 \phi \phi^* + \frac{1}{4} F_{kl} F_{kl} \ , \tag{11}$$

consequently, the canonical Hamiltonian is $H_C = \int d^3y \ \mathcal{H}_C$, with $\int d^3y = \int dy^1 dy^2 dy^-$.

We also define the primary Hamiltonian H_P adding to the canonical Hamiltonian the primary constraints with their respective Lagrange multipliers

$$H_P = H_C + \int d^3y \left(\mathrm{w}_1 C + \mathrm{u}_k \chi^k + \mathrm{v}^* \Gamma + \Gamma^* \mathrm{v}\right), \tag{12}$$

where w_1, u_k are the multipliers related to the electromagnetic constraints and, v and v^* are the respective multipliers for the scalar constraints.

The fundamental Poisson brackets (PB) between the fields are

$$\left\{A_\mu \left(x\right), \pi^\mu \left(y\right)\right\} = \delta^\nu_\mu \delta^3 \left(x - y\right), \tag{13}$$

$$\left\{\phi \left(x\right), \pi^* \left(y\right)\right\} = \delta^3 \left(x - y\right) \qquad , \qquad \left\{\phi^* \left(x\right), \pi \left(y\right)\right\} = \delta^3 \left(x - y\right) \quad . \tag{14}$$

The Dirac's procedure tell us that the primary constraints must be preserved in time (consistence condition) under time evolution generated by the primary Hamiltonian by requiring that they have a weakly vanishing PB with H_P. Thus, such requirement on the scalar constraints yields

$$\begin{aligned} \dot{\Gamma} &= -ig\left[\phi \pi^- + 2D_- \left(A_+ \phi\right)\right] - D^k D_k \phi - m^2 \phi - 2D_- \mathrm{v} \approx 0, \\ \dot{\Gamma}^* &= ig\left[\phi^* \pi^- + 2D_-^* \left(A_+ \phi^*\right)\right] - \left(D_k D^k \phi\right)^* - m^2 \phi^* - 2\left(D_- \mathrm{v}\right)^* \approx 0. \end{aligned} \tag{15}$$

These relations determine the multipliers v and v, respectively, and there are not more constraints associate with the scalar sector.

In the electromagnetic sector, the consistence condition of χ^k yields

$$\dot{\chi}^k = \partial_k \pi^- + j^k + \partial_j F_{jk} - 2\partial_- \mathrm{u}_k \approx 0, \tag{16}$$

an equation for its associated multiplier u_k.

Finally, the consistence condition of π^+ gives a secondary constraint G

$$\dot{C} = G = \partial_- \pi^- + \partial_k \pi^k + j^+ \approx 0, \tag{17}$$

which is simply the Gauss's law. It is easy to verify that there are not further constraints generate from the consistence condition of the Gauss's law because it is automatically conserved

$$\dot{G} \equiv ig\left[\phi^* \dot{\Gamma} - \phi \dot{\Gamma}^*\right] \approx 0. \tag{18}$$

Then, there are not more constraints in the theory and the full set of constraints given by (9), (10) and (17).

A. Constraint classification

The constraint π^+ has vanishing PB with all the other constraints therefore it is a first class constraint. The remaining set $\Phi^a = \left\{\Gamma,\ \Gamma^*,\ G,\ \chi^k\right\}$ is apparently a second class set, however, the determinant of its constraint matrix $\left\{\Phi^a\left(x\right), \Phi^b\left(y\right)\right\}$ is zero (see Appendix C) because the matrix has a zero mode whose eigenvector gives a linear combination of constraints which is one more first class constraint. Alternately, there is an additional argument, the constraint G, the Gauss's law, is the second first class constraint for zero coupling constant: the free Maxwell's field theory. On the other hand, if G belongs to a minimal set of second class constraints, the limit for zero coupling constant would no longer be possible because the DB would become undefined. Remembering that the DB is defined with respect to a non-singular matrix, however, it could become singular when we go back to the free theory. In conclusion, there is a linear combination which is independent of π^+ and it is a first class constraint. Such that linear combination is

$$\Sigma \equiv G - ig\left(\phi^*\Gamma - \phi\Gamma^*\right). \tag{19}$$

Thus, we have the following set of second class constraints

$$\begin{aligned} \Gamma &\equiv \pi - D_-\phi \approx 0 \qquad , \qquad \Gamma^* \equiv \pi^* - \left(D_-\phi\right)^* \approx 0 \quad , \\ \chi^k &\equiv \pi^k - \partial_- A_k + \partial_k A_- \approx 0 \end{aligned} \tag{20}$$

and the set of first class constraints

$$C = \pi^+ \approx 0 \qquad , \qquad \Sigma = G - ig\left(\phi^*\Gamma - \phi\Gamma^*\right) \approx 0. \tag{21}$$

We can be sure that is the maximal number of first class constraints since the quest for time independence of $\left(\Gamma,\ \Gamma^*,\ \chi^k\right)$ leads to equations that determine their respective Lagrange multipliers. The second first class constraint in (19), must be contrasted with the *instant form* analysis [9] where the second first class constraint is not a linear combination of electromagnetic and scalar constraints because in this formalism the scalar sector does not have any constraint.

B. Equations of motion and gauge fixing conditions

At this point we need to check that we have the correct (Euler-Lagrange) equations of motion. Thus, the time derivative of the fields is calculated by computing their PB with the so called extended Hamiltonian (H_E) what now it generates the time translations or temporal evolutions. The H_E is obtained by adding to the primary Hamiltonian H_P all the first class constraints, thus, we get

$$\begin{aligned} H_E &= \int d^3y \left[\frac{1}{2}\left(\pi^-\right)^2 + \left(\pi^-\partial_- + \pi^k\partial_k - j^+\right)A_+ - D_k\phi\left(D^k\phi\right)^* + m^2\phi\phi^* \right. \\ &\left. + \frac{1}{4}F_{kl}F_{kl}\right] + \int d^3y \left[\, \mathrm{w}_1 C + \mathrm{u}_k\chi^k + \mathrm{v}^*\Gamma + \Gamma^*\mathrm{v} + \mathrm{w}_2\Sigma\right]. \end{aligned} \tag{22}$$

Thus we have that the time evolution of the dynamical variables of the electromagnetic field are

$$\begin{aligned} &\dot{A}_+ = \mathrm{w}_1 && , \qquad \dot{\pi}^+ = G \approx 0 \\ &\dot{A}_- = \pi^- + \partial_- A_+ - \partial_-\mathrm{w}_2 && , \qquad \dot{\pi}^- = 2g^2\phi\phi^* A_+ + \partial_k \mathrm{u}_k + ig\mathrm{v}^*\phi - ig\phi^*\mathrm{v} \\ &\dot{A}_k = \partial_k A_+ + \mathrm{u}_k - \partial_k \mathrm{w}_2 && , \qquad \dot{\pi}^k = ig\phi\left(D^k\phi\right)^* - ig\phi^* D^k\phi + \partial_j F_{jk} - \partial_- \mathrm{u}_k \end{aligned}$$

and for the scalar fields are given by

$$\begin{aligned} \dot{\phi} &= \mathrm{v} + ig\mathrm{w}_2\phi \\ \dot{\phi}^* &= \mathrm{v}^* - ig\mathrm{w}_2\phi^* \\ \dot{\pi}^* &= -\left(D_-D_+\phi\right)^* + igA_+\left(D_-\phi\right)^* - \left(D_kD^k\phi\right)^* - m^2\phi^* - ig\phi^* D_-^*\mathrm{w}_2 \\ & \quad -2ig\mathrm{w}_2\left(D_-\phi\right)^* \\ \dot{\pi} &= -D_-D_+\phi - igA_+D_-\phi - D^kD_k\phi - m^2\phi + ig\phi D_-\mathrm{w}_2 + 2ig\mathrm{w}_2 D_-\phi. \end{aligned}$$

From which, it is easy to obtain

$$\begin{aligned} D_\mu D^\mu \phi + m^2\phi &\approx ig\phi D_- \mathrm{w}_2 + 2ig\mathrm{w}_2 D_-\phi \,, \\ \left(D_\mu D^\mu \phi\right)^* + m^2\phi^* &\approx -ig\phi^* D_-^* \mathrm{w}_2 - 2ig\mathrm{w}_2 \left(D_-\phi\right)^* \,, \\ \partial_\mu F^{\mu\nu} + j^\nu &\approx 0 \,, \end{aligned} \tag{23}$$

thus, the equation of motions are consistent with its Lagrangian form if we choose $\mathrm{w}_2 = 0$.

The Dirac's algorithm requires as many gauge conditions as first class constraints there are. However, such gauge fixing conditions must be compatible with the Euler-Lagrange equations and therefore they must fix the Lagrangian multiplier w_2 to zero and together with the first class constraints must be a second class set. One set of gauge conditions satisfying such requirements is

$$A_- \approx 0 \qquad , \qquad \pi^- + \partial_- A_+ \approx 0 \tag{24}$$

which are standard in the pure gauge theory, the so-called null-plane gauge [7, 10].

C. Dirac's brackets (DB)

We follow the iterative method to calculate the Dirac's brackets, thus, we first consider the set of the first class constraints (21) and their gauge fixing conditions (24)

$$\begin{aligned} &\Phi_1 \equiv \pi^+ && , \quad \Phi_3 \equiv A_- \\ &\Phi_2 \equiv \Sigma = G - ig\left(\phi^*\Gamma - \phi\Gamma^*\right) && , \quad \Phi_4 \equiv \pi^- + \partial_- A_+, \end{aligned} \tag{25}$$

such that the set of constraints (25) is second class and whose constraint matrix $C_{ij}\left(x,y\right) \equiv \left\{\Phi_i\left(x\right), \Phi_j\left(y\right)\right\}$, with components:

$$C_{ij}\left(x,y\right) = \begin{pmatrix} 0 & 0 & 0 & \partial_-^x \\ 0 & 0 & -\partial_-^x & 0 \\ 0 & -\partial_-^x & 0 & 1 \\ \partial_-^x & 0 & -1 & 0 \end{pmatrix} \delta^3\left(x-y\right). \tag{26}$$

must be regular. However, in order to solve the inverse of the constraint matrix (26) we require a suitable inverse for the longitudinal derivative ∂_-. In general, the operator ∂_- has the following inverse:

$$\left(\partial_-\right)^{-1} f(x^-) = \frac{1}{2}\int dy^- \, \epsilon\left(x^- - y^-\right) f\left(y^-, x^+, x^\perp\right) + F\left(x^+, x^\perp\right), \tag{27}$$

where the function $\epsilon\left(x\right)$ is

$$\epsilon\left(x\right) = \begin{cases} 1 \;, & x > 0 \\ 0 \;, & x = 0 \\ -1 \;, & x < 0 \end{cases} \tag{28}$$

and $F\left(x^+, x^\perp\right)$ is a x^- arbitrary independent function. The presence of F implies that constraint matrix (26) does not have a unique inverse, nevertheless, Dirac proved that the matrix formed by a complete set of second class constraints should be unique, therefore, it is said that the set of second class constraints in (25) is not purely second class.

Steinhardt [7] proved that the inverse matrix of (26) is not unique because among the second class constraints there are a hidden subset first class constraints [8]. This subset of constraints can be evidenced by observing that the most general solution for (15) and (16) is:

$$\begin{aligned} \mathrm{v}\left(x\right) &= \hat{\mathrm{v}}\left(x\right) + s\left(x^+, x^\perp\right), \\ \mathrm{v}^*\left(x\right) &= \hat{\mathrm{v}}^*\left(x\right) + s^*\left(x^+, x^\perp\right), \\ \mathrm{u}_k\left(x\right) &= \hat{\mathrm{u}}_k\left(x\right) + s_k\left(x^+, x^\perp\right), \end{aligned} \tag{29}$$

where $s\left(x^{+},x^{\perp}\right)$, $s^{*}\left(x^{+},x^{\perp}\right)$ and $s_{k}\left(x^{+},x^{\perp}\right)$ are arbitrary functions of $\left(x^{+},x^{\perp}\right)$, and $\hat{\mathrm{v}}\left(x\right)$, $\hat{\mathrm{v}}^{*}\left(x\right)$ and $\hat{\mathrm{u}}_{k}\left(x\right)$ represent the "unambiguous" solutions. Now, if we insert the v, v^{*} and u_{k} of (29) into the extended Hamiltonian (22), then

$$
\begin{aligned}
H_{E}^{\prime} \;=\; & \int d^{3}y\left[\frac{1}{2}\left(\pi^{-}\right)^{2}+\left(\pi^{-}\partial_{-}+\pi^{k}\partial_{k}-j^{+}\right)A_{+}-D_{k}\phi\left(D^{k}\phi\right)^{*}+m^{2}\phi\phi^{*}\right. \\
& \left.+\frac{1}{4}F_{kl}F_{kl}\right]+\int d^{3}y\left[\,\mathrm{w}_{1}C+\hat{\mathrm{u}}_{k}\chi^{k}+\hat{\mathrm{v}}^{*}\Gamma+\Gamma^{*}\hat{\mathrm{v}}+\mathrm{w}_{2}\Sigma\right] \\
& +\int d^{3}y\left[\,s_{k}\chi^{k}+s^{*}\Gamma+\Gamma^{*}s\right].
\end{aligned}
\tag{30}
$$

Thus, despite that the set of constraints (25) seem to be second class, the multipliers v, v^{*} and u_{k} are not completely fixed implying that the Hamiltonian still contains the arbitrary functions s, s^{*} and s_{k}.

Steinhardt shown that this hidden subset of first class constraints is associated with improper gauge transformations [7]. An improper gauge transformation can not identifies with generators of gauge transformations [8], as occurs with the first class constraints (21). Since this kind of constraints are tied to boundary conditions, they can map a given physical solution to another one with different boundary conditions, which is no equivalent to the former [8] [9]. Therefore, it is not possible to eliminate the improper gauge transformations by means of gauge conditions since such procedure would exclude configurations physically allowed to the system. This hidden constraints can be eliminated by fixing appropriated boundary conditions on the fields in order to the total Hamiltonian be a true generator of time evolution of the physical system.

Thus, in order to evaluate explicitly the inverse of the matrix of second class constraints (26) and ensure its uniqueness we must determine $F\left(x^{+},x^{\perp}\right)$. This function can be evaluated if we impose the appropriate boundary conditions on the fields (ϕ, ϕ^{*}, A_{k}) given in the reference [5]. Under such boundary conditions, the inverse of the operator ∂_{-} is defined on all integrable functions $f(x^{-})$ which are less singular than $\frac{1}{x^{-}}$ and vanish faster than $\frac{1}{x^{-}}$ for large x^{-}, namely:

$$
\left(\partial_{-}\right)^{-1}f(x^{-})=\frac{1}{2}\int dy^{-}\epsilon\left(x^{-}-y^{-}\right)f(y^{-}).
\tag{31}
$$

With this, we get a unique inverse of (26) which is given by

$$
C_{ij}^{-1}\left(x,y\right)\equiv\frac{1}{2}\begin{pmatrix} 0 & -\left|x^{-}-y^{-}\right| & 0 & \epsilon\left(x^{-}-y^{-}\right) \\ \left|x^{-}-y^{-}\right| & 0 & -\epsilon\left(x^{-}-y^{-}\right) & 0 \\ 0 & -\epsilon\left(x^{-}-y^{-}\right) & 0 & 0 \\ \epsilon\left(x^{-}-y^{-}\right) & 0 & 0 & 0 \end{pmatrix}\delta^{2}\left(x^{\perp}-y^{\perp}\right).
\tag{32}
$$

Alternatively, we can determine the inverse by insisting that the DB satisfy Jacobi identities [11], given the same result.

Using the inverse defined by equation (32), the first set of DB, $\{\cdot,\cdot\}_{D1}$, for given two dynamical variables $\mathbf{A}\left(x\right)$ and $\mathbf{B}\left(y\right)$ are calculate by

$$
\begin{aligned}
\left\{\mathbf{A}\left(x\right),\mathbf{B}\left(y\right)\right\}_{D1} \;=\; & \left\{\mathbf{A}\left(x\right),\mathbf{B}\left(y\right)\right\}-\int d^{3}ud^{3}v\left\{\mathbf{A}\left(x\right),\Phi_{i}\left(u\right)\right\} \\
& C_{ij}^{-1}\left(u,v\right)\left\{\Phi_{j}\left(v\right),\mathbf{B}\left(y\right)\right\}.
\end{aligned}
\tag{33}
$$

Thus, the no null DB_{1} are

$$
\begin{aligned}
\left\{\phi\left(x\right),A_{+}\left(y\right)\right\}_{D1} &= \frac{ig}{2}\phi\left(x\right)\left|x^{-}-y^{-}\right|\delta^{2}\left(x^{\perp}-y^{\perp}\right), \\
\left\{\phi^{*}\left(x\right),A_{+}\left(y\right)\right\}_{D1} &= -\frac{ig}{2}\phi^{*}\left(x\right)\left|x^{-}-y^{-}\right|\delta^{2}\left(x^{\perp}-y^{\perp}\right), \\
\left\{A_{k}\left(x\right),A_{+}\left(y\right)\right\}_{D1} &= -\frac{1}{2}\left|x^{-}-y^{-}\right|\partial_{k}^{x}\delta^{2}\left(x^{\perp}-y^{\perp}\right).
\end{aligned}
\tag{34}
$$

Now, following with the iterative procedure to calculate DB [9], we consider the subset of the remaining second class constraints that under the brackets DB_1 are given as

$$\begin{array}{ll} \Psi_1 \equiv \Gamma = \pi - \partial_- \phi \quad , & \Psi_2 \equiv \Gamma^* = \pi^* - \partial_- \phi^* \\ \Psi_3 \equiv \chi^1 \quad , & \Psi_4 \equiv \chi^2 \end{array} , \tag{35}$$

where $\chi^k = \pi^k - \partial_- A_k$. The constraint matrix from this set is defines as

$$D_{ij}\left(x,y\right) \equiv \left\{ \Psi_i\left(x\right), \Psi_j\left(y\right) \right\}_{D1} . \tag{36}$$

Using the boundary conditions on the fields accepted to calculate (32), we compute the inverse D^{-1} and thus we define the second set of DB, $\{\cdot,\cdot\}_{D2}$,

$$\begin{aligned} \left\{ \mathbf{A}\left(x\right), \mathbf{B}\left(y\right) \right\}_{D2} &= \left\{ \mathbf{A}\left(x\right), \mathbf{B}\left(y\right) \right\}_{D1} - \int d^3u d^3v \left\{ \mathbf{A}\left(x\right), \Psi_i\left(u\right) \right\}_{D1} \\ & \qquad D^{-1}_{ij}\left(u,v\right) \left\{ \Psi_j\left(v\right), \mathbf{B}\left(y\right) \right\}_{D1} . \end{aligned} \tag{37}$$

Then, we obtain the final DB among the fundamental dynamical variables of the theory

$$\begin{aligned} \left\{ A_k\left(x\right), A_l\left(y\right) \right\}_{D2} &= -\frac{1}{4}\delta^k_l \epsilon\left(x^- - y^-\right) \delta^2\left(x^\perp - y^\perp\right) , \\ \left\{ \phi\left(x\right), \phi^*\left(y\right) \right\}_{D2} &= -\frac{1}{4}\epsilon\left(x^- - y^-\right) \delta^2\left(x^\perp - y^\perp\right) , \end{aligned} \tag{38}$$

$$\begin{aligned} \left\{ \phi\left(x\right), A_+\left(y\right) \right\}_{D2} &= \frac{i}{2} g\phi\left(x\right) \left|x^- - y^-\right| \delta^2\left(x^\perp - y^\perp\right) - \frac{i}{8} g\delta^2\left(x^\perp - y^\perp\right) \\ & \int dv^- \ \epsilon\left(x^- - v^-\right) \phi\left(x^\perp, v^-\right) \epsilon\left(v^- - y^-\right) , \end{aligned} \tag{39}$$

$$\begin{aligned} \left\{ \phi^*\left(x\right), A_+\left(y\right) \right\}_{D2} &= -\frac{i}{2} g\phi^*\left(x\right) \left|x^- - y^-\right| \delta^2\left(x^\perp - y^\perp\right) + \frac{i}{8} g\delta^2\left(x^\perp - y^\perp\right) \\ & \int dv^- \ \epsilon\left(x^- - v^-\right) \phi^*\left(x^\perp, v^-\right) \epsilon\left(v^- - y^-\right) . \end{aligned} \tag{40}$$

From the correspondence principle we obtain the following commutators among the fields

$$\begin{aligned} \left[A_k\left(x\right), A_l\left(y\right) \right] &= -\frac{i}{4}\delta^k_l \epsilon\left(x^- - y^-\right) \delta^2\left(x^\perp - y^\perp\right) , \\ \left[\phi\left(x\right), \phi^*\left(y\right) \right] &= -\frac{i}{4}\epsilon\left(x^- - y^-\right) \delta^2\left(x^\perp - y^\perp\right) , \end{aligned} \tag{41}$$

$$\begin{aligned} \left[\phi\left(x\right), A_+\left(y\right) \right] &= -\frac{1}{2} g\phi\left(x\right) \left|x^- - y^-\right| \delta^2\left(x^\perp - y^\perp\right) + \frac{1}{8} g\delta^2\left(x^\perp - y^\perp\right) \\ & \int dv^- \ \epsilon\left(x^- - v^-\right) \phi\left(x^\perp, v^-\right) \epsilon\left(v^- - y^-\right) , \end{aligned} \tag{42}$$

$$\begin{aligned} \left[\phi^*\left(x\right), A_+\left(y\right) \right] &= \frac{1}{2} g\phi^*\left(x\right) \left|x^- - y^-\right| \delta^2\left(x^\perp - y^\perp\right) - \frac{1}{8} g\delta^2\left(x^\perp - y^\perp\right) \\ & \int dv^- \ \epsilon\left(x^- - v^-\right) \phi^*\left(x^\perp, v^-\right) \epsilon\left(v^- - y^-\right) . \end{aligned} \tag{43}$$

The first two relations were specified by Neville and Rohrlich [12]. They derived this expressions starting from the free field operators commutation relations for unequal times x^+, y^+ and then, they calculated the commutators on the null-plane at equal time, *i. e.*, $x^+ = y^+$. The commutation relations involving the field operator $\hat{A}_+(x)$ were not obtained in [12] but it was affirmed that they must be derived solving a quantum constraint. However, we get to show that it is possible to obtain these last commutation relations at classical level following a carefully Dirac's analysis of the constraint structure of the model.

III. SPINOR ELECTRODYNAMICS (QED_4): CONSTRAINT STRUCTURE

The Lagrangian density of the spinor Electrodynamics written in terms of the light cone projections[18] of the fermionic fields is

$$\begin{aligned}\mathcal{L} \;=\; & \bar{\psi}_+\left(\frac{i}{2}\gamma^+\overleftrightarrow{\partial}_+ - gA_+\gamma^+\right)\psi_+ + \bar{\psi}_-\left(\frac{i}{2}\gamma^-\overleftrightarrow{\partial}_- - gA_-\gamma^-\right)\psi_- \\ & +\bar{\psi}_+\left(\frac{i}{2}\overleftrightarrow{\not\partial} - g\not{A} - m\right)\psi_- + \bar{\psi}_-\left(\frac{i}{2}\overleftrightarrow{\not\partial} - g\not{A} - m\right)\psi_+ \\ & -\frac{1}{2}\left(F_{12}\right)^2 + \frac{1}{2}\left(F_{+-}\right)^2 + F_{+k}F_{-k}\ ,\end{aligned} \tag{44}$$

where we have defined $\gamma^k A_k = \not{A}$ for $k = 1, 2$. The corresponding field equations are:

$$\begin{aligned}\partial_\nu F^{\nu\mu} - g\bar{\psi}\gamma^\mu\psi &= 0 \\ (i\partial_+ - gA_+)\gamma^+\psi_+ + [i\not\partial - g\not{A} - m]\psi_- &= 0 \\ (i\partial_- - gA_-)\gamma^-\psi_- + [i\not\partial - g\not{A} - m]\psi_+ &= 0 \\ (i\partial_+ + gA_+)\bar{\psi}_+\gamma^+ + \bar{\psi}_-\left[i\overleftarrow{\not\partial} + g\not{A} + m\right] &= 0 \\ (i\partial_- + gA_-)\bar{\psi}_-\gamma^- + \bar{\psi}_+\left[i\overleftarrow{\not\partial} + g\not{A} + m\right] &= 0.\end{aligned} \tag{45}$$

The canonical momenta for the fields are

$$\pi^\mu = F^{\mu +} = \partial^\mu A^+ - \partial^+ A^\mu \tag{46}$$

and

$$\begin{aligned}&\bar{\phi}_{+a} = -\frac{i}{2}\bar{\psi}_{+b}\left(\gamma^+\right)_{ba} \quad , \quad \phi_{+a} = -\frac{i}{2}\left(\gamma^+\right)_{ab}\psi_{+b}\ , \\ &\bar{\phi}_{-a} = 0 \quad , \quad \phi_{-a} = 0\ ,\end{aligned} \tag{47}$$

where $a, b = 1, 2, 3, 4$.

From the canonical momenta equations, we observe that the only one equation in (46) is dynamical

$$\pi^- = F^{-+} = \partial_+ A_- - \partial_- A_+, \tag{48}$$

and all the other equations coming from (46) and (47) give a set of primary constrains composite by three bosonic constraints

$$C \equiv \pi^+ \approx 0 \quad , \quad C^k \equiv \pi^k + \partial_k A_- - \partial_- A_k \approx 0 \quad , \quad k = 1, 2 \tag{49}$$

and four fermionic constraints

$$\begin{aligned}&\Gamma_{+a} \equiv \phi_{+a} + \tfrac{i}{2}\left(\gamma^+\right)_{ab}\psi_{+b} \approx 0 \quad , \quad \bar{\Gamma}_{+a} \equiv \bar{\phi}_{+a} + \tfrac{i}{2}\bar{\psi}_{+b}\left(\gamma^+\right)_{ba} \approx 0 \\ &\Gamma_{-a} \equiv \phi_{-a} \approx 0 \quad , \quad \bar{\Gamma}_{-a} \equiv \bar{\phi}_{-a} \approx 0\ .\end{aligned} \tag{50}$$

The canonical Hamiltonian density is [13]

$$\begin{aligned}\mathcal{H}_c \;=\; & \frac{1}{2}\left(\pi^-\right)^2 + \left[\pi^-\partial_- + \pi^k\partial_k + g\bar{\psi}_+\gamma^+\psi_+\right]A_+ + \frac{1}{2}\left(F_{12}\right)^2 \\ & -\bar{\psi}_-\left[\frac{i}{2}\gamma^-\overleftrightarrow{\partial}_- - gA_-\gamma^-\right]\psi_- - \bar{\psi}_+\left[\frac{i}{2}\overleftrightarrow{\not\partial} - g\not{A} - m\right]\psi_- \\ & -\bar{\psi}_-\left[\frac{i}{2}\overleftrightarrow{\not\partial} - g\not{A} - m\right]\psi_+,\end{aligned} \tag{51}$$

and the primary Hamiltonian takes the form

$$H_P = H_c + \int dy^3 \left[uC + u_k C^k + \bar{\Gamma}_{+a} v_{1a} + \bar{\Gamma}_{-a} v_{2a} - \bar{v}_{1a} \Gamma_{+a} - \bar{v}_{2a} \Gamma_{-a} \right] , \tag{52}$$

where u and u_k are bosonic Lagrange multipliers and, v_1, v_2, $\bar{v}_1$ and $\bar{v}_2$ are fermionic multipliers.

The fundamental Poisson brackets are

$$\{A_\mu(x), \pi^\nu(y)\} = \delta_\mu^\nu \delta^3(x-y) , \tag{53}$$

$$\left\{\psi_{+a}(x), \bar{\phi}_{+b}(y)\right\} = -\delta_{ab}\delta^3(x-y) \quad , \quad \left\{\bar{\psi}_{+a}(x), \phi_{+b}(y)\right\} = -\delta_{ab}\delta^3(x-y) ,$$

$$\left\{\psi_{-a}(x), \bar{\phi}_{-b}(y)\right\} = -\delta_{ab}\delta^3(x-y) \quad , \quad \left\{\bar{\psi}_{-a}(x), \phi_{-b}(y)\right\} = -\delta_{ab}\delta^3(x-y) .$$

Next, we give the non null PB's between the primary constraints

$$\begin{aligned} \left\{\Gamma_{+a}(x), \bar{\Gamma}_{+b}(y)\right\} &= -i\left(\gamma^+\right)_{ab} \delta^3(x-y) \ , \\ \left\{\bar{\Gamma}_{+a}(x), \Gamma_{+b}(y)\right\} &= -i\left(\gamma^+\right)_{ba} \delta^3(x-y) \ , \\ \left\{C^k(x), C^j(y)\right\} &= -2\delta_j^k \partial_-^x \delta^3(x-y) \ . \end{aligned} \tag{54}$$

A. The fermionic sector

In order to the primary constraints be preserved in the time, we compute the consistence condition of the fermionic constraints (50). For Γ_+ we obtain

$$\dot{\Gamma}_+ = \left(i\partial\!\!\!/ - gA\!\!\!/ - m\right)\psi_- - gA_+\gamma^+\psi_+ - i\gamma^+ v_1 \approx 0 \ , \tag{55}$$

from this equation we can get two relations by using the null-plane γ-algebra. First we do $\frac{i}{2}\gamma^-\dot{\Gamma}_+$ then we get one component of the multiplier v_1

$$\Delta^+ v_1 = -\frac{i}{2}\gamma^-\left(i\partial\!\!\!/ - gA\!\!\!/ - m\right)\psi_- + igA_+\psi_+. \tag{56}$$

The second relation is obtained by the projection $\Delta^+\dot{\Gamma}_+$ getting a secondary constraint

$$\Delta^+\dot{\Gamma}_+ = \left(i\partial\!\!\!/ - gA\!\!\!/ - m\right)\Delta^+\psi_- \approx 0 \ , \tag{57}$$

due to $\left(i\partial\!\!\!/ - gA\!\!\!/ - m\right)$ is an invertible operator we can set the secondary constraint as being

$$\Phi = \Delta^+\psi_- \approx 0 \ . \tag{58}$$

For $\bar{\Gamma}_+$, we get

$$\dot{\bar{\Gamma}}_+ = \bar{\psi}_-\left(i\overleftarrow{\partial\!\!\!/} + gA\!\!\!/ + m\right) + gA_+\bar{\psi}_+\gamma^+ - i\bar{v}_1\gamma^+ \approx 0 \ . \tag{59}$$

From $\frac{i}{2}\dot{\bar{\Gamma}}_+\gamma^-$ we get one component of the multiplier $\bar{v}_1$

$$\bar{v}_1\Delta^- = -\frac{i}{2}\bar{\psi}_-\left(i\overleftarrow{\partial\!\!\!/} + gA\!\!\!/ + m\right)\gamma^- - igA_+\bar{\psi}_+ \ \ , \tag{60}$$

and from $\dot{\bar{\Gamma}}_+\Delta^-$ we get another secondary constraint

$$\dot{\bar{\Gamma}}_+\Delta^- = \bar{\psi}_-\Delta^-\left(i\overleftarrow{\partial\!\!\!/} + gA\!\!\!/ + m\right) \approx 0 \ , \tag{61}$$

similarly, $\left(i\overleftarrow{\partial\!\!\!/} + gA\!\!\!/ + m\right)$ is an invertible operator, thus, we set this secondary constraint to be

$$\bar{\Phi} = \bar{\psi}_-\Delta^- \approx 0. \tag{62}$$

The consistence condition of the constraints Γ_- and $\bar{\Gamma}_-$ we get more two secondary constraints

$$\Omega_- = \dot{\Gamma}_- = \gamma^- \left(i\partial_- - gA_-\right)\psi_- + \left(i\partial\!\!\!/ \; - gA\!\!\!/ - m\right)\psi_+ \approx 0 \tag{63}$$

and

$$\bar{\Omega}_- = \dot{\bar{\Gamma}}_- = \left(i\partial_- + gA_-\right)\bar{\psi}_-\gamma^- + \bar{\psi}_+ \left(i\overleftarrow{\partial\!\!\!/} + gA\!\!\!/ + m\right) \approx 0 \tag{64}$$

The consistence condition of the secondary constraints Φ, $\bar{\Phi}$, Ω_- and $\bar{\Omega}_-$ gives relations for some components of the fermionic multipliers, as we will show. Thus, the conservation in time of Φ and $\bar{\Phi}$ fixes one projection for each multiplier v_2 and $\bar{v}_2$

$$\dot{\Phi} = -\Delta^+ v_2 \approx 0 \qquad , \qquad \dot{\bar{\Phi}} = -\bar{v}_2\Delta^- \approx 0. \tag{65}$$

And the conservation in time of Ω_- and $\bar{\Omega}_-$ gives two equations relating the multipliers v_2, v_1 and $\bar{v}_2$, $\bar{v}_1$, respectively,

$$\dot{\Omega}_- = -\gamma^- \left(i\partial_- - gA_-\right) v_2 - \left(i\partial\!\!\!/ \; - gA\!\!\!/ - m\right) v_1 \approx 0, \tag{66}$$

and,

$$\dot{\bar{\Omega}}_- = -\left(i\partial_- + gA_-\right)\bar{v}_2\gamma^- - \bar{v}_1 \left(i\overleftarrow{\partial\!\!\!/} + gA\!\!\!/ + m\right) \approx 0. \tag{67}$$

At once, we will show that the set of equation (56), (60), (65), (66) and (67) allows to define completely all the fermionic Lagrange multipliers. Thus, doing $\Delta^-\dot{\Omega}_-$ we get the other component for v_1

$$\Delta^- v_1 = 0 \ , \tag{68}$$

then using that $\Delta^+ v_1 + \Delta^- v_1 = v_1$, we determine the multiplier v_1

$$v_1 = -\frac{i}{2}\gamma^- \left(i\partial\!\!\!/ - gA\!\!\!/ - m\right)\psi_- + igA_+\psi_+ \ . \tag{69}$$

Also in (66) doing $\frac{1}{2}\gamma^+\dot{\Omega}_-$ we determinate the component $\Delta^- v_2$ by the equation

$$\left(i\partial_- - gA_-\right)\left(\Delta^- v_2\right) = \frac{i}{2}\left[-\left(i\partial\!\!\!/ - gA\!\!\!/\right)^2 + m^2\right]\psi_- - \frac{i}{2}gA_+\gamma^+ \left(i\partial\!\!\!/ \; - gA\!\!\!/ - m\right)\psi_+ \tag{70}$$

the joined to component $\Delta^+ v_2$ in (65) gives

$$\left(i\partial_- - gA_-\right) v_2 = \frac{i}{2}\left[-\left(i\partial\!\!\!/ - gA\!\!\!/\right)^2 + m^2\right]\psi_- - \frac{i}{2}gA_+\gamma^+ \left(i\partial\!\!\!/ \; - gA\!\!\!/ - m\right)\psi_+. \tag{71}$$

A similar procedure allows determinate the multipliers $\bar{v}_1$ and $\bar{v}_2$

$$\bar{v}_1 = -\frac{i}{2}\bar{\psi}_- \left(i\overleftarrow{\partial\!\!\!/} + gA\!\!\!/ + m\right)\gamma^- - igA_+\bar{\psi}_+ \ , \tag{72}$$

and

$$\left(i\partial_- + gA_-\right)\bar{v}_2 = \frac{i}{2}\bar{\psi}_- \left[-\left(i\overleftarrow{\partial\!\!\!/} + gA\!\!\!/\right)^2 + m^2\right] + \frac{i}{2}gA_+\bar{\psi}_+ \left(i\overleftarrow{\partial\!\!\!/} + gA\!\!\!/ + m\right)\gamma^+ \ , \tag{73}$$

respectively. Therefore, all the fermionic Lagrange multipliers were determined, then, the set of primary and secondary fermionic constraints are second class according to Dirac's procedure [9]. The use of the projection of the fermionic fields permitted to observe clearly the existence of fermionic secondary constraints which show that the field is fully describe by only two of their four components.

B. The electromagnetic sector

The consistent condition of C^k gives

$$\dot{C}^k = \left(\delta_2^k \partial_1 - \delta_1^k \partial_2\right) F_{12} - g\left(\bar{\psi}_+ \gamma^k \psi_- + g\bar{\psi}_- \gamma^k \psi_+\right) + \partial_k \pi^- - 2\partial_- u_k \approx 0, \tag{74}$$

it is a differential equation for the u_k Lagrange multipliers.

The consistence condition of π^+ produces the following secondary constraint

$$G \equiv \dot{\pi}^+ = \partial_- \pi^- + \partial_k \pi^k - g\bar{\psi}_+ \gamma^+ \psi_+ \approx 0, \tag{75}$$

which is the Gauss's law, and its consistence condition shows that

$$\dot{G} = ig\left(\dot{\bar{\Gamma}}_+ \psi_+ + \bar{\psi}_+ \dot{\Gamma}_+ + \bar{\psi}_- \dot{\Gamma}_- + \dot{\bar{\Gamma}}_- \psi_-\right) \approx 0 \ , \tag{76}$$

what is automatically conserved in time. Then, no more constraints in the theory are generated and the multiplier u relative to π^+ remains undetermined.

C. Constraint classification and gauge fixing conditions

The full set of primary and secondary constraints is given by the equations (49), (50), (58), (62), (63), (64) and (75)

$$C \quad , \quad C^k \quad , \quad \Gamma_+ \quad , \quad \Gamma_- \quad , \quad \bar{\Gamma}_+ \quad , \quad \bar{\Gamma}_- \quad , \quad \Omega_- \quad , \quad \Phi \quad , \quad \bar{\Omega}_- \quad , \quad \bar{\Phi} \quad , \quad G \ . \tag{77}$$

The $C = \pi^+$ has a vanishing PB with each one of the constraints and therefore it is a first class constraint. Apparently, the remaining subset of constraints is second class, but they form a singular constraint matrix with an zero mode whose respective eigenvector gives a linear combination what is one more first class constraint (see Appendix **C**). Alternately, we must observe that as the fermionic case, the electromagnetic sector must maintain its free constraint structure due that the interaction term is not allowed to change the first class structure of the free theory into second class ones, because the DB would not be defined any longer in the limit of zero coupling constant. Thus, such combination, which is independent of π^+ and it is a first class constraint, is

$$\Sigma \equiv G - ig\left[\bar{\psi}_+ \Gamma_+ + \bar{\Gamma}_+ \psi_+ + \bar{\psi}_- \Gamma_- + \bar{\Gamma}_- \psi_-\right] . \tag{78}$$

Thus, we have the following set of second class constraints

$$\Gamma_+ \quad , \quad \Gamma_- \quad , \quad \Omega_- \quad , \quad \Phi \quad , \quad \bar{\Gamma}_- \quad , \quad \bar{\Gamma}_+ \quad , \quad \bar{\Omega}_- \quad , \quad \bar{\Phi} \quad , \quad C^k \tag{79}$$

and the set of first class constraints

$$C \quad , \quad \Sigma \ . \tag{80}$$

This is the maximal number of first class constraints and the consistence condition on the second class constraints led to expressions for their respective Lagrange multipliers.

Now, the next step is to impose gauge conditions one for every first class constraint, such that the set of gauge fixing conditions and first class constraints turn on a second class set. The choosing of the appropriate set of gauge conditions is a careful procedure, because they should be compatible with the Euler-Lagrange equations of motion. Thus, we choose a set of gauge conditions known as the null-plane gauge and it is defined by the following relations

$$B = A_- \approx 0 \quad , \quad K \equiv \pi^- + \partial_- A_+ \approx 0 \tag{81}$$

then, the set of first class constraints and gauge fixing conditions now **is** a second class set.

It in worthwhile to note that when the photon field is couple with the fermion field, we would consider A_+ as a possible gauge condition but in this case it is not possible to find a second gauge condition to be compatible with the equations of motion. It is similar what happen with the radiation gauge in the instant form formalism, *i. e.*, $x^0 = cte.$ plane [9].

D. The Dirac's brackets

The explicit evaluation of the inverse of the full matrix of second class constraint involves an arbitrary function, which can be determined considering appropriated boundary conditions on the fields [5], thus, the inverse of the operator ∂_- is defined as in (31).

After a laborious work, we obtain the graded Lie algebra for the dynamical variables of the spinor electrodynamics

$$\{A_k(x), A_j(y)\}_D = -\frac{1}{4}\delta_j^k \epsilon\left(x^- - y^-\right)\delta^2\left(x^\perp - y^\perp\right), \tag{82}$$

$$\begin{aligned}\left\{\psi_a(x), \bar{\psi}_b(y)\right\}_D \;=\; & -\frac{i}{2}\left(\gamma^-\right)_{ab}\delta^3(x-y) - \frac{1}{4}\,\epsilon\left(x^- - y^-\right) \\ & \left(i\not{D}_\perp^x + m\right)_{ab}\delta^2\left(x^\perp - y^\perp\right) + \frac{i}{8}\left|x^- - y^-\right| \\ & \left\{\gamma^+\left[\left(\not{D}_\perp^x\right)^2 + m^2\right]\right\}_{ab}\delta^2\left(x^\perp - y^\perp\right),\end{aligned} \tag{83}$$

$$\begin{aligned}\{A_+(x), \psi(y)\}_D \;=\; & -\frac{i}{2}g\left|x^- - y^-\right|\delta^2\left(x^\perp - y^\perp\right)\psi(y) \\ & +\frac{i}{4}g\delta^2\left(x^\perp - y^\perp\right)\int dz^- \epsilon\left(x^- - z^-\right)\epsilon\left(z^- - y^-\right)\Delta^-\psi\left(x^\perp, z^-\right) \\ & +\frac{i}{16}g\delta^2\left(x^\perp - y^\perp\right)\int dz^-\left|x^- - z^-\right|\epsilon\left(z^- - y^-\right)\left[\gamma^+\gamma^k\partial_k^x\psi\left(x^\perp, z^-\right)\right] \\ & +\frac{i}{16}g\left[\partial_k^x\delta^2\left(x^\perp - y^\perp\right)\right]\int dz^-\left|x^- - z^-\right|\epsilon\left(z^- - y^-\right)\left[\gamma^+\gamma^k\psi\left(x^\perp, z^-\right)\right],\end{aligned} \tag{84}$$

$$\begin{aligned}\left\{A_+(x), \bar{\psi}(y)\right\}_D \;=\; & \frac{i}{2}g\left|x^- - y^-\right|\delta^2\left(x^\perp - y^\perp\right)\bar{\psi}(y) \\ & -\frac{i}{4}g\delta^2\left(x^\perp - y^\perp\right)\int dz^- \epsilon\left(x^- - z^-\right)\epsilon\left(z^- - y^-\right)\bar{\psi}\left(x^\perp, z^-\right)\Delta^+ \\ & -\frac{i}{16}g\left[\partial_j^x\delta^2\left(x^\perp - y^\perp\right)\right]\int dz^-\left|x^- - z^-\right|\epsilon\left(z^- - y^-\right)\left[\bar{\psi}\left(x^\perp, z^-\right)\gamma^j\gamma^+\right] \\ & -\frac{i}{16}g\delta^2\left(x^\perp - y^\perp\right)\int dz^-\left|x^- - z^-\right|\epsilon\left(z^- - y^-\right)\left[\partial_k^x\bar{\psi}\left(x^\perp, z^-\right)\gamma^k\gamma^+\right].\end{aligned} \tag{85}$$

Our two first expressions, equations (82) and (83), are in accord with the result obtained by Rohrlich [5] and Kogut and Soper [14] when the correspondence principle is applied. In [5] these relations are determined from the (anti)-commutator between the field operators for unequal times while in [14] the authors postulated the (anti)-commutators for the annihilation and creation operators for the quanta fields, and transforming these relations back to coordinate space, they obtained the equal-time (anti)-commutators.

The relations (84) and (85) associated with the field A_+ were derived following a careful application of the Dirac's procedure to the null-plane. Using the correspondence principle these relations are equivalents with the expressions derived by Kogut and Soper [14].

IV. REMARKS AND CONCLUSIONS

We have performed the constraint analysis of the scalar and spinor electrodynamics on the null-plane and several characteristics or features are in contrast with the customary space-like hyper-surface formulation.

We have shown that the $SQED_4$ has a first class constraint, the Gauss's law, which result of a linear combination of electromagnetic and scalar constraints which is given by the zero mode eigenvector of the constraint matrix. This fact is a consequence of the constraints associated with the scalar sector. On the other hand, in the instant form analysis the second first class constraint does not have contributions coming from scalar constraints because in this formalism the scalar sector is free of constraints.

After select the null-plane gauge conditions to transform the first class constraints in second class one, we need to impose appropriated boundary conditions on the fields to fix a hidden subset of first class constraints which allows to get an unique inverse of the second class constraints matrix. Then, we obtain the DB's of the theory and can quantize it via the correspondence principle. The commutation relations among fields (41) obtained by us are consistent with those results reported in the literature [12]. The relations (43) involving the field operator A_+ were not obtained in [12] but it was affirmed that they could be derived solving a quantum constraint. However, these commutators were calculated by us by quantizing the DB's derived at classical level following a careful analysis of the constraints structure of the $SQED_4$.

In QED_4 case, the careful analysis the constraints structure of the fermionic sector shows that it has only second class constraints. However, there exists a hidden subset of first class constraints [7] which generate improper gauge transformations [8]. Such first class subset are associated with the impossibility of define an unique inverse for the operator ∂_- related to the insufficiency of boundary conditions to solve the Cauchy data problem. The uniqueness of the inverse is guaranteed by imposing appropriated boundary conditions on the fields.

The first class set of the QED_4 are fixed choosing the null-plane gauge conditions and following the Dirac's procedure we obtain graded algebra (82)-(85) for the canonical variables. Via the correspondence principle (82) and (83) reproduce the canonical (anti)-commutation relations for the quantum fields derived in [5, 14]. Also the relations (84) and (85) associated with the field A_+, derived following a careful application of the Dirac's procedure, reproduce the expressions derived by Kogut and Soper [14] at quantum level.

Recently [17] the null-plane Hamiltonian structure of the (1+1)-dimensional electrodynamics, the Schwinger model, has been studied by following the Dirac's procedure [2] and dealing carefully the hidden first class constraints [7, 8]. The study shows that the fermionic sector has only second class constraints such as happened in the instant formalism. And as we can be shown the fermionic sector of the QED_4 in the null-plane also presents only second class constraints structure. It can be conclude that fermionic fields satisfying a first order Dirac's equation in the null-plane or instant form formalisms have only second class constraints structure.

In advanced, we are studying the constraints structure analysis of the pure Yang-Mills fields and also the Hamiltonian structure of the theory resulting of the interaction between the Yang-Mils fields with complex scalar fields. Reports on this research will be communicated elsewhere.

Acknowledgements

BMP thanks CNPq for partial support. GERZ thanks CNPq (grant 142695/2005-0) for full support.

Appendix A: Notation

The null plane time x^+ and longitudinal coordinate x^- are defined, respectively, as

$$x^+ \equiv \frac{x^0 + x^3}{\sqrt{2}} \qquad x^- \equiv \frac{x^0 - x^3}{\sqrt{2}}, \tag{A1}$$

with the transverse coordinates $x^\perp \equiv (x^1, x^2)$ kept unchanged.

Hence, in the space of four-vectors $x = (x^+, x^1, x^2, x^-)$, the metric is

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{A2}$$

Explicitly,

$$x^+ = x_- \ , \quad x^- = x_+ \ \ , \quad x \cdot y = x^+ y^- + x^- y^+ - x^\perp \cdot y^\perp \ \ , \tag{A3}$$

where the derivatives with respect to x^+ e x^- are defined as

$$\partial_+ \equiv \frac{\partial}{\partial x^+} \ , \qquad \partial_- \equiv \frac{\partial}{\partial x^-} \tag{A4}$$

with $\partial^+ = \partial_-$. Here, we have used the following relations

$$\delta^4(x-y) = \delta(x^+ - y^+)\delta^2(x^\perp - y^\perp)\delta(x^- - y^-).$$

$$\frac{1}{2}\frac{d}{dx^-}\epsilon(x^- - y^-) = \delta(x^- - y^-) \quad , \quad \frac{1}{2}\int dy^- \, \epsilon(x^- - y^-)\epsilon(y^- - z^-) = |x^- - y^-| \tag{A5}$$

The same orthogonal transformation is applied to Dirac matrices that still obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{A6}$$

this makes γ^+ and γ^- singular matrices.

Since

$$(\gamma^+)^\dagger = \gamma^- \ , \qquad (\gamma^-)^\dagger = \gamma^+ \qquad (\gamma^k)^\dagger = -\gamma^k \qquad k = 1, 2. \tag{A7}$$

we define the Hermitian matrices

$$\Delta^\pm = \frac{1}{2}\gamma^\mp\gamma^\pm \ , \tag{A8}$$

which are projection operators,

$$(\Delta^\pm)^2 = \Delta^\pm \ , \qquad \Delta^\pm\Delta^\mp = 0 \ , \qquad \Delta^+ + \Delta^- = 1 \quad . \tag{A9}$$

Their action on Dirac spinors yields

$$\psi_\pm = \Delta^\pm\psi \ , \qquad \bar{\psi}_\pm = \bar{\psi}\Delta^\mp, \tag{A10}$$

Appendix B: The second first class constraint for SQED and QED

Now, we compute the second first class constraint for $SQED_4$. The set of the remaining constraints is $\Phi^a = \left\{\Gamma, \ \Gamma^*, \ G, \ \chi^k\right\}$, its constraint matrix $C_{ab}(x,y) = \{\Phi^a(x), \Phi^a(y)\} = \Delta(x)\,\delta^3(x-y)$, where the $\Delta(x)$ matrix is

$$\Delta(x) = \begin{pmatrix} 0 & -2\left(D_-^x\right)^* & 2ig\left[\left(D_-^x\right)^*\phi(x)\right] + 2ig\phi(x)\,\partial_-^x & 0 \\ -2D_-^x & 0 & -2ig\left[D_-^x\phi^*(x)\right] - 2ig\phi^*(x)\,\partial_-^x & 0 \\ 2ig\phi(x)\,D_-^x & -2ig\phi^*(x)\left(D_-^x\right)^* & F(x) & 0 \\ 0 & 0 & 0 & -2\delta_l^k\partial_-^x \end{pmatrix} \tag{B1}$$

where the operator F is

$$F \equiv -2g^2\partial_-(\phi\phi^*) - 4g^2\phi\phi^*\partial_-. \tag{B2}$$

The matrix $C(x,y)$ has determinant zero. It is due to this matrix has a zero eigenvalue and its respective eigenvector gives a linear combination of the constraints which is a first class constraint. The eigenvector is calculated using the following equation

$$\int d^3y \ C_{ab}(x,y)\,U_b(y) = 0, \tag{B3}$$

thus, the eigenvector is $U = (-ig\phi^*, \ ig\phi, \ 1, \ 0, \ 0)^T$ and the linear combination given the second first class is

$$\Sigma = \Phi^a U_a = G - ig\left(\phi^*\Gamma - \phi\Gamma^*\right). \tag{B4}$$

For the QED_4, the remaining set of constraints is

$$G \quad , \quad \Gamma_+ \quad , \quad \Gamma_- \quad , \quad \bar{\Gamma}_+ \quad , \quad \bar{\Gamma}_- \quad , \quad \Omega_- \quad , \quad \Phi \quad , \quad \bar{\Omega}_- \quad , \quad \bar{\Phi} \quad , \quad C^k. \tag{B5}$$

Its constraint matrix has a sole zero mode and the respective eigenvector is

$$U = (1, \ ig\bar{\psi}_+, \ ig\bar{\psi}_-, \ -ig\psi_+, -ig\psi_-, \ 0, \ 0, \ 0, \ 0, \ 0, \ 0)^T, \tag{B6}$$

it gives the second first class

$$\Sigma = G - ig\left[\bar{\psi}_+\Gamma_+ + \bar{\Gamma}_+\psi_+ + \bar{\psi}_-\Gamma_- + \bar{\Gamma}_-\psi_-\right]. \tag{B7}$$

Appendix C: Grassmann Algebras

A Grassmann algebra contains bosonic (self-commuting) and fermionic (self-anticommuting) variables [15]:

$$FB = (-1)^{n_A n_B} BF \quad , \tag{C1}$$

where $n = 0$ for a bosonic, and $n = 1$ for a fermionic variable. Note that the product of two fermionic variables is bosonic, and the product of a fermionic and a bosonic variables is fermionic.

The left derivative of a ψ_α fermionic variable is defined as

$$\frac{\partial}{\partial \psi_\alpha}\left\{ \psi_{\alpha_1}\psi_{\alpha_2} \ \cdots \psi_{\alpha_n} \right\} = -\delta_{\alpha\alpha_1}\psi_{\alpha_2} \ \cdots \psi_{\alpha_n} + \delta_{\alpha\alpha_2}\psi_{\alpha_1}\psi_{\alpha_3} \ \cdots \psi_{\alpha_n} + \ \cdots \ + (-1)^n \delta_{\alpha\alpha_n}\psi_{\alpha_1}\psi_{\alpha_2} \ \cdots \psi_{\alpha_{n-1}} \ . \tag{C2}$$

The Poisson Brackets can be defined similar to ordinary mechanics [16]. The phase space is spanned by q_i, p^i which are bosons and ψ_α and π^α, fermions. Denote by B(F) a bosonic (Fermionic) element of the Grassmann algebra, then

$$\begin{aligned} \{B_1, B_2\} = -\{B_2, B_1\} &= \left\{ \frac{\partial B_1}{\partial q_i}\frac{\partial B_2}{\partial p^i} - \frac{\partial B_2}{\partial q_i}\frac{\partial B_1}{\partial p^i} \right\} + \left\{ \frac{\partial B_1}{\partial \phi_\alpha}\frac{\partial B_2}{\partial \pi^\alpha} - \frac{\partial B_2}{\partial \phi_\alpha}\frac{\partial B_1}{\partial \pi^\alpha} \right\} \\ \{F, B\} = -\{B, F\} &= \left\{ \frac{\partial F}{\partial q_i}\frac{\partial B}{\partial p^i} - \frac{\partial B}{\partial q_i}\frac{\partial F}{\partial p^i} \right\} - \left\{ \frac{\partial F}{\partial \phi_\alpha}\frac{\partial B}{\partial \pi^\alpha} + \frac{\partial B}{\partial \phi_\alpha}\frac{\partial F}{\partial \pi^\alpha} \right\} \\ \{F_1, F_2\} = \{F_2, F_1\} &= \left\{ \frac{\partial F_1}{\partial q_i}\frac{\partial F_2}{\partial p^i} + \frac{\partial F_2}{\partial q_i}\frac{\partial F_1}{\partial p^i} \right\} - \left\{ \frac{\partial F_1}{\partial \phi_\alpha}\frac{\partial F_2}{\partial \pi^\alpha} + \frac{\partial F_2}{\partial \phi_\alpha}\frac{\partial F_1}{\partial \pi^\alpha} \right\} \quad . \end{aligned} \tag{C3}$$

It follow from its definition that the Poisson brackets has the properties

$$\begin{aligned} \{A, B\} &= -(-1)^{n_A n_B}\{B, A\} \\ \{A, B + C\} &= \{A, B\} + \{A, C\} \\ \{A, BC\} &= (-1)^{n_A n_B} B\{A, C\} + \{A, B\}C \\ \{AB, C\} &= (-1)^{n_B n_C}\{A, C\}B + A\{B, C\} \\ (-1)^{n_A n_C}\{A, \{B, C\}\} &+ (-1)^{n_B n_A}\{B, \{C, A\}\} + (-1)^{n_C n_B}\{C, \{A, B\}\} = 0 \quad . \end{aligned} \tag{C4}$$

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