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THE SCHWINGER MODEL ON THE NULL-PLANE

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We study the Schwinger Model on the null-plane using the Dirac method for constrained systems. The fermion field is analyzed using the natural null-plane projections coming from the γ-algebra and it is shown that the fermionic sector of the Schwinger Model has only second class constraints. However, the first class constraints are exclusively of the bosonic sector. Finally, we establish the graded Lie algebra between the dynamical variables, via generalized Dirac bracket in the null-plane gauge, which is consistent with every constraint of the theory.

Keywords: null-plane; Hamiltonian method; Schwinger model.

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1. Introduction

Half the of last century Dirac[1] proposed three different forms of relativistic dynamics depending on the types of surfaces where independent modes were initiated. One of them is the *front form*, which is a surface of a single light wave, commonly referred to as *null-plane* (*light-front or light-cone*) formalism. A notable feature of a relativistic theory on the null-plane is that it gives rises to a constrained dynamical system.[2] Srivastava[3] studied the light-front quantization of the bosonized version of the Schwinger model in the continuum formalism, the propose of his work was to show that the quantization of the massless Schwinger model on the light-front

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leads in a straightforward way to the θ-vacua structure. Eller and Pauli[4] applied the method of discretized light-cone quantization to the case of massive and massless electrons, obtaining the correct mass for the Schwinger particle and reproducing correctly many known features of the spectrum.

The aim of the present work is to construct the Hamiltonian formulation of the Schwinger model in the null-plane description and to obtain a graded algebra among the fundamental dynamical variables of the theory. The constraint analysis shows the existence of hidden first class constraints[5] and we are going to show that when we impose the appropriate boundary conditions[6] on the fields we eliminate this hidden first class constraints[7]; showing that the constraint analysis in Ref. 8 is wrong. The work is organized as follow. In the Sec. 2 we are going to do the constraint analysis, next we classify the constraints and we impose the corresponding gauge fixing conditions. Finally, we invert the constraints by imposing appropriate boundary conditions and the Dirac brackets (DB) of the theory are calculated. In the last section, we summarize the results obtained by us.

2. Massive Schwinger Model

The gauge theory we are considering is defined by the following Lagrangian density

$$\begin{aligned}\mathcal{L} = &\frac{i}{2}\bar{\varphi}_+\gamma^+\partial_+\varphi_+ - \frac{i}{2}\partial_+\bar{\varphi}_+\gamma^+\varphi_+ + \frac{i}{2}\bar{\varphi}_-\gamma^-\partial_-\varphi_- - \frac{i}{2}\partial_-\bar{\varphi}_-\gamma^-\varphi_- + \\ &- m\bar{\varphi}_+\varphi_- - m\bar{\varphi}_-\varphi_+ - gA_+\bar{\varphi}_+\gamma^+\varphi_+ - gA_-\bar{\varphi}_-\gamma^-\varphi_- - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \quad (1)\end{aligned}$$

the coordinate x^+ will be our time variable and $\varphi_\pm$, $\bar{\varphi}_\pm$ are the fermionic projections in the light-cone system (For notations see Ref. 8).

From the relations defining the conjugate momenta we obtain the following primary constraints

$$C \equiv \pi^+ \approx 0, \qquad \Gamma_+ \equiv p_+ + \frac{i}{2}\gamma^+\psi_+ \approx 0, \qquad \Gamma_- \equiv p_- \approx 0 \qquad (2)$$

$$\bar{\Gamma}_+ \equiv \bar{p}_+ + \frac{i}{2}\bar{\psi}_+\gamma^+ \approx 0, \qquad \bar{\Gamma}_- \equiv \bar{p}_- \approx 0\,. \qquad (3)$$

Also we get the dynamical relation $\pi^- = \partial_+A_- - \partial_-A_+$. Also, $\pi^\pm$ is the momentum conjugate to $A_\pm$, $\bar{p}_\pm$ to $\varphi_\pm$ and $p_\pm$ to $\bar{\varphi}_\pm$, respectively.

The canonical Hamiltonian density is

$$\begin{aligned}\mathcal{H}_c = &\frac{1}{2}\left(\pi^-\right)^2 + \pi^-\partial_-A_+ - \frac{i}{2}\bar{\varphi}_-\gamma^-\partial_-\varphi_- + \frac{i}{2}\partial_-\bar{\varphi}_-\gamma^-\varphi_- + \\ &+ gA_-\bar{\varphi}_-\gamma^-\varphi_- + gA_+\bar{\varphi}_+\gamma^+\varphi_+ + m\bar{\varphi}_+\varphi_- + m\bar{\varphi} \qquad (4)\end{aligned}$$

and, $H_P = H_c + \int dy^-\left[uC + \bar{\Gamma}_+v_1 + \bar{\Gamma}_-v_2 - \bar{v}_1\Gamma_+ - \bar{v}_2\Gamma_-\right]$, is the primary Hamiltonian. u is a bosonic Lagrange multiplier and $v_1, v_2, \bar{v}_1, \bar{v}_2$ are fermionic multipliers.

The consistence condition on the fermionic constraints yields the following set of secondary constraints

$$\chi = \Delta^{+}\psi_{-} \approx 0, \quad \Omega_{-} = \gamma^{-}\left(i\partial_{-} - gA_{-}\right)\psi_{-} - m\psi_{+} \approx 0 \tag{5}$$

$$\bar{\chi} = \bar{\psi}_{-}\Delta^{-} \approx 0, \quad \bar{\Omega}_{-} = \left(i\partial_{-} + gA_{-}\right)\bar{\psi}_{-}\gamma^{-} + m\bar{\psi}_{+} \approx 0\,, \tag{6}$$

and equations for some components of the fermionic multipliers. The preservation under time evolution of the secondary fermionic constraints time results in additional conditions on the fermionic multipliers being they determined completely. In the similar way the consistence of the bosonic primary constraints yields $\dot{C} = \partial_{-}\pi^{-} - g\bar{\varphi}\gamma^{+}\varphi \equiv G \approx 0$, which is a secondary constraint, named as Gauss's law, where its consistence condition shows that it is automatically conserved in time, then no more constraints in the theory are generated.

2.1. *Constraint classification*

Now, we are going to classify the constraints. It is clear to shows that π^{+} is a first-class constraint. The remaining subset $\{G, \Gamma_{+}, \Gamma_{-}, \chi, \Omega_{-}, \bar{\Gamma}_{+}, \bar{\Gamma}_{-}, \bar{\chi}, \bar{\Omega}_{-}\}$ has a singular constraint matrix and therefore it can be shown that it has only one zero mode whose eigenvector gives a following first class constraint,[9]

$$\Sigma = G - ig\left[\bar{\psi}_{+}\Gamma_{+} + \bar{\Gamma}_{+}\psi_{+} + \bar{\psi}_{-}\Gamma_{-} + \bar{\Gamma}_{-}\psi_{-}\right]. \tag{7}$$

Then, the subset of first class constraints is π^{+} and Σ and the subset of fermionic second-class constraints is $\left\{\Gamma_{+},\ \Gamma_{-},\ \chi,\ \Omega_{-},\ \bar{\Gamma}_{+},\ \bar{\Gamma}_{-},\ \bar{\chi},\ \bar{\Omega}_{-}\right\}$. Our result clarifies and it corrects the result found in Ref. 8, where the constraint analysis affirms the existence of proper first class constraints in the fermionic sector, but, as we have just shown such a statement it is not true when a careful analysis is carried out. However, it is possible to show that the first class nature in the fermionic sector is related to the hidden subset of first-class constraints which generate improper gauge transformations[5] associated with the insufficiency of the initial value data.

Now the next step is to impose gauge conditions, one for every first class constraint such that the set of gauge fixing conditions and first class constraints turns on a second class set. The choosing of the appropriate set of gauge conditions is a careful procedure, because they should be compatible with the Euler-Lagrange equations of motion. Thus, we choose a set of gauge conditions known as the null-plane gauge and it is defined by the following relations $B = A_{-} \approx 0, K \equiv \pi^{-} + \partial_{-}A_{+} \approx 0$, which are standard in the pure gauge theory.[10]

2.2. *Dirac brackets*

To obtain the Dirac brackets is necessary the explicit evaluation of the inverse of the matrix of second class constraints; but this inverse is not unique, it involves an arbitrary function which is related to the hidden first class constraints mentioned previously. However, this hidden first class subset can be fixed by considering appropriated boundary conditions[6] on the fields $(\varphi, \bar{\varphi}, A_{\mu})$, and then a unique inverse for the constraint matrix is obtained. Thus, the DB among the fundamental variables of the theory are

$$\{\varphi_a(x), \bar{\varphi}_b(y)\}_D = \frac{im^2}{8}\gamma^+_{ab}\ |x-y| - \frac{m}{4}\mathbb{I}_{ab}\,\epsilon\,(x-y) - \frac{i}{2}\gamma^-_{ab}\ \delta\,(x-y) \tag{8}$$

$$\{\varphi\,(x)\,, A_+\,(y)\}_D = i\frac{g}{2}\varphi\,(x)\,|x-y| - i\frac{g}{4}\int dv\ \epsilon\,(x-v)\,\Delta^-\varphi\,(v)\,\epsilon\,(\ v-y) \tag{9}$$

$$\{\bar{\varphi}\,(x)\,, A_+\,(y)\}_D = -\frac{ig}{2}\bar{\varphi}\,(x)\,|x-y| + i\frac{g}{4}\int dv\,\epsilon\,(x-v)\,\bar{\varphi}\,(v)\,\Delta^+\epsilon\,(v-y) \tag{10}$$

3. Remarks and Conclusion

We have performed the constraint analysis of the (1+1) dimensional massive QED and the careful analysis of the fermionic sector shows that it has only second class constraints and, the first class constraints are exclusive of the electromagnetic sector. The fermionic second class constraints allow to show the fermionic fields are fully described by only one of their two components.

In the Ref. 8, the constraints analysis follows an erroneous procedure and it gets to show the existence of proper first class constraint in the fermionic sector. Such affirmations can be easily drop down by considering that the Dirac equation is a linear equation of first order in time in the front form formalism where the fermionic sector presents a second class structure. When we pass to the null-plane formalism the Dirac equation remains of first order in time, therefore, the constraint classification must be second class, too. As we show, the first class constraints for the fermionic sector reported in Ref. 8 do not exist in the sense of proper ones, however, a type of improper first class constraint is related to the ambiguity in the definition of the inverse of the operator ∂_-, or in other words, they are related to the zero modes of the operator. Such ambiguities are eliminated by fixing the necessary boundary conditions. Then, our contribution presents a correct use of the Dirac procedure applied to null-plane field theories.

Finally, choosing the light-cone gauge we fix the bosonic first class constraints and the graded Lie algebra for the canonical variables is given via generalized Dirac brackets. The graded algebra for massless Schwinger model can be obtained from the massive case doing the limit when $m \to 0$ in Eqs. (8)–(10).

The obtained graded algebra via the correspondence principle reproduce the canonical (anti)-commutation relations obtained at quantum level in Ref. 6.

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