Stability and Symmetry Breaking in the Two-Higgs-Doublet Model

Yithsbey Giraldo U.

Instituto de Física
Universidad de Antioquia
Universidad de Nariño
Medellín, Colombia
22 de Mayo.
[hep-ph/0605184]
Motivations

- **Standard Model (SM)** contains one Higgs doublet \( \varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \)
  
  - potential \( V_{SM} = -\mu^2 \varphi^\dagger \varphi + \lambda (\varphi^\dagger \varphi)^2 \).
  
  - after symmetry breaking: \( 4-3 = 1 \) real d.o.f \( \cong \) 1 Higgs boson.

- motivations for extended Higgs sector:
  - supersymmetry, baryogenesis, . . .

- **Two-Higgs-Doublet Model (THDM)** as “simplest ext.”:
  \( \varphi_1 = \begin{pmatrix} \varphi_1^+ \\ \varphi_0^0 \\ \varphi_1 \end{pmatrix} \), \( \varphi_2 = \begin{pmatrix} \varphi_2^+ \\ \varphi_2^0 \end{pmatrix} \)
  
  - potential more involved
  
  - after symmetry breaking: \( 8-3 = 5 \) real d.o.f \( \cong \) 1 charged pair, 3 neutral Higgs boson.

- literatura on THDMs: huge amount (specific models, recently basis independent methods)

- here: stability and symmetry breaking in *most general* THDM at tree level (constructions for arbitrary basis)
Outline of the talk

1. THDM Potential
   - orbit variables
Outline of the talk

1. THDM Potential
   - orbit variables

2. Stability
   - Criteria for Stability
   - Example
Outline of the talk

1. THDM Potential
   - orbit variables

2. Stability
   - Criteria for Stability
   - Example

3. Symmetry Breaking
   - Stationary Points
   - Criteria for Symmetry Breaking
   - Example
How can we describe the most general THDM?

- two complex Higgs-doublet fields with hypercharge $y = +1/2$:
  \[
  \varphi_1(x) = \begin{pmatrix} \varphi_1^+(x) \\ \varphi_1^0(x) \end{pmatrix}, \quad \varphi_2(x) = \begin{pmatrix} \varphi_2^+(x) \\ \varphi_2^0(x) \end{pmatrix}.
  \]

- renormalisable, gauge invariant potential contains only
  \[
  \varphi_i^\dagger \varphi_j, \quad (\varphi_i^\dagger \varphi_j)(\varphi_k^\dagger \varphi_l), \quad i, j, k, l \in \{1, 2\}
  \]

Definition: orbit variables $K_0, K_1, K_2, K_3$:

\[
\begin{pmatrix}
\varphi_1^\dagger \varphi_1 & \varphi_2^\dagger \varphi_1 \\
\varphi_1^\dagger \varphi_2 & \varphi_2^\dagger \varphi_2
\end{pmatrix} \equiv \frac{1}{2} (K_0 \mathbb{1} + K_a \sigma^a) \Leftrightarrow \begin{cases}
K_0 = \varphi_1^\dagger \varphi_1 + \varphi_2^\dagger \varphi_2, & K_1 = 2 \text{Re} \varphi_1^\dagger \varphi_2, \\
K_3 = \varphi_1^\dagger \varphi_1 - \varphi_2^\dagger \varphi_2, & K_2 = 2 \text{Im} \varphi_1^\dagger \varphi_2
\end{cases}
\]

- general THDM Higgs potential:
  \[
  V(\varphi_1, \varphi_2) = V_2 + V_4 \quad \text{with} \quad \begin{cases}
V_2 = \xi_0 K_0 + \xi_a K_a \\
V_4 = \eta_{00} K_0^2 + 2 K_0 \eta_a K_a + K_a \eta_{ab} K_b
\end{cases}
  \]

- no gauge d.o.f. in this scheme, reduced powers
Orbit Variables

- **domain** \((\mathbf{K} \equiv (K_1 \ K_2 \ K_3)^T)\):

\[
K_0 = ||\varphi_1||^2 + ||\varphi_2||^2 \geq 0 \\
K_0^2 - K^2 = 4 \left( ||\varphi_1||^2 ||\varphi_2||^2 - |\varphi_1^\dagger \varphi_2|^2 \right) \geq 0
\]

- **change of doublet basis** by \(U \in U(2)\)

\[
\begin{pmatrix}
\varphi'_1 \\
\varphi'_2
\end{pmatrix} = U 
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix},
\]

means for orbit variables

\[
K'_0 = K_0, \quad \mathbf{K}' = R(U)\mathbf{K}, \quad R(U) \in SO(3), \quad U^\dagger \sigma^a U = R_{ab}(U)\sigma^b.
\]

- **Minkowski type structure**:\((K_0, \mathbf{K})\) on and inside "forward light cone".

\[
(K_2 = 0)
\]
Stability

- stable potential: bounded from below
- stability determined by \( V \) in limit \(|\varphi_1|^2 + |\varphi_2|^2 = K_0 \to \infty\),
- consider \( V_4 \) for \( K_0 > 0 \), define:

\[
k \equiv K/K_0, \quad \text{with } |k| \leq 1.
\]

\[
V_4 = K_0^2 J_4(k), \quad J_4(k) \equiv \eta_{00} + 2\eta^T k + k^T E k
\]

- stability guaranteed by \( V_4 \iff J_4(k) > 0 \) for all \(|k| \leq 1\)
- domain of \( J_4 \) is unit ball: compact
  \( J_4 > 0 \iff J_4|_{stat} > 0 \) for all its stationary points
- stationary points of \( J_4(k) \equiv \eta_{00} + 2\eta^T k + k^T E k \) on domain \( k^2 \leq 1 \)

\[
|k| < 1: \text{solve } \nabla_k J_4(k) = 0 \iff Ek = -\eta \quad \text{with } 1 - k^2 > 0
\]

\[
|k| = 1: \text{define } F_4(k, u) \equiv J_4(k) + u(1 - k^2) \quad \text{with Lagrange multiplier } u,
\]

\[
\text{solve } \nabla_k F_4(k, u) = 0 \iff (E - u)k = -\eta \quad \text{with } 1 - k^2 = 0
\]
Stability

Stability criteria via one function

unified description with

\[ f(u) \equiv F_4(k(u), u) = u + \eta_{00} - \eta^T (E - u)^{-1} \eta \]

for regular stationary points (set \( u = 0 \) for \( |k| < 1 \)):

\[ f(u) = J_4(k) = V_4(k)/K_0^2, \]

\[ f'(u) = 1 - k^2 \]

define \( I = \) “set of all \( u \) values belonging to stat. pnts. of \( J_4(k) \)”:

\[ I := \{ u | f'(u) = 0 \lor \]

\[ u = 0 \land f'(0) > 0 \} \]

( exceptional solutions omitted here)

Theorem

Stability of potential guaranteed by \( V_4 \iff f(u_i) > 0 \) for all \( u_i \in I \)
Illustration of Stability Determining Function

stability determined by
\[ f(u) \equiv F_4(k(u), u) \]

explicitely
\[
\begin{align*}
    f(u) &= u + \eta_{00} - \eta^T (E - u)^{-1} \eta, \\
    f'(u) &= 1 - \eta^T (E - u)^{-2} \eta,
\end{align*}
\]

in a basis where \( E = \text{diag}(\mu_1, \mu_2, \mu_3) \):
\[
\begin{align*}
    f(u) &= u + \eta_{00} - \sum_{a=1}^{3} \frac{\eta_a^2}{\mu_a - u}, \\
    f'(u) &= 1 - \sum_{a=1}^{3} \frac{\eta_a^2}{(\mu_a - u)^2}.
\end{align*}
\]

\( f'(u) \) has at most 6 zeros

( exceptional solutions only possible if corresponding \( \eta_a = 0 \) )

figure: \( f'(u), f(u) \) for \( \eta_{00} = 5 \cdot 10^{-2} \), \( \eta_a = 2 \cdot 10^{-3} \), \( (\mu_1, \mu_2, \mu_3) = (1, 2, 3) \cdot 10^{-2} \)
Example: THDM of Gunion et al.

Potential

Higgs potential:

\[ V = \lambda_1 (\varphi_1^+ \varphi_1 - v_1^2)^2 + \lambda_2 (\varphi_2^+ \varphi_2 - v_2^2)^2 + \lambda_3 (\varphi_1^+ \varphi_1 - v_1^2 + \varphi_2^+ \varphi_2 - v_2^2)^2 \]
\[ + \lambda_4 ((\varphi_1^+ \varphi_1)(\varphi_2^+ \varphi_2) - (\varphi_1^+ \varphi_2)(\varphi_2^+ \varphi_1)) \]
\[ + \lambda_5 (\text{Re}(\varphi_1^+ \varphi_2) - v_1 v_2 \cos \xi)^2 + \lambda_6 (\text{Im}(\varphi_1^+ \varphi_2) - v_1 v_2 \sin \xi)^2 \]
\[ + \lambda_7 (\text{Re}(\varphi_1^+ \varphi_2) - v_1 v_2 \cos \xi)(\text{Im}(\varphi_1^+ \varphi_2) - v_1 v_2 \sin \xi) \]

\( V \) breaks \((\varphi_1, \varphi_2) \leftarrow (-\varphi_1, \varphi_2)\) only softly

\( V_4 \) parameters:

\[ \eta_{00} = \frac{1}{4} (\lambda_1 + \lambda_2 + 4\lambda_3 + \lambda_4), \]
\[ \eta = \frac{1}{4} \begin{pmatrix} 0 \\ 0 \\ \lambda_1 - \lambda_2 \end{pmatrix}, \quad E = \frac{1}{8} \begin{pmatrix} 2(\lambda_5 - \lambda_4) & \lambda_7 & 0 \\ \lambda_7 & 2(\lambda_6 - \lambda_4) & 0 \\ 0 & 0 & 2(\lambda_1 + \lambda_2 - \lambda_4) \end{pmatrix}. \]
Example: THDM of Gunion et al.

Stability

stability guaranteed by $V_4^{\text{theorem}}$

\[ \lambda_1 + \lambda_3 > 0, \quad \lambda_2 + \lambda_3 > 0, \quad \lambda_4, \kappa > -2\lambda_3 - 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \]

where $\kappa := \frac{1}{2}(\lambda_5 + \lambda_6 - \sqrt{(\lambda_5 - \lambda_6)^2 + \lambda_7^2})$. 
Stationary Points

- 4-vector notation: \( \tilde{K} = \begin{pmatrix} K_0 \\ K \end{pmatrix} \), \( \tilde{\xi} = \begin{pmatrix} \xi_0 \\ \xi \end{pmatrix} \), \( \tilde{E} = \begin{pmatrix} \eta_{00} & \eta^T \\ \eta & E \end{pmatrix} \)

- potential \( V = \tilde{K}^T \tilde{\xi} + \tilde{K}^T \tilde{E}\tilde{K} \)

- three classes of stationary points:
  - \( K_0 = K = 0 \):
    - Trivial solution (\( \varphi_1 = \varphi_2 = 0 \))
  - \( K_0 > K \): solve (\( \nabla_{\tilde{K}} V = 0 \))
  - \( K_0 = K > 0 \): solve (\( \nabla_{\tilde{K},u} [V - u(K_0^2 - K^2)] = 0 \))
Stationary Points

What are the implications for EWSB?

Consider symmetry breaking behaviour of
\[ \langle \varphi_1 \rangle \equiv \begin{pmatrix} v_1^+ \\ v_1^0 \end{pmatrix}, \quad \langle \varphi_2 \rangle \equiv \begin{pmatrix} v_2^+ \\ v_2^0 \end{pmatrix} \]

for three global minimum \( K_0, K \) cases

\[
K_0 = ||\varphi_1||^2 + ||\varphi_2||^2 \geq 0
\]

\[
K_0^2 - K^2 = 4(||\varphi_1||^2||\varphi_2||^2 - |\varphi_1 \varphi_2|^2 \geq 0
\]

- \( K_0 = K = 0 \) \( \Rightarrow \) \( \langle \varphi_1 \rangle = \langle \varphi_2 \rangle = 0 \) \( \Rightarrow \) unbroken \( SU(2)_L \otimes U(1)_Y \)
- \( K_0 > |K| \) \( \Rightarrow \) \( \langle \varphi_1 \rangle, \langle \varphi_2 \rangle \) lin. indep. \( \Rightarrow \) fully broken \( SU(2)_L \otimes U(1)_Y \)
- \( K_0 = |K| > 0 \) \( \Rightarrow \) \( \langle \varphi_1 \rangle, \langle \varphi_2 \rangle \) lin. dep. \( \Rightarrow \) \( SU(2)_L \otimes U(1)_Y \rightarrow U(1)_{em} \)
Stationary Points

- 4-vector notation: \( \tilde{\mathbf{K}} = \begin{pmatrix} K_0 \\ \mathbf{K} \end{pmatrix} \), \( \tilde{\xi} = \begin{pmatrix} \xi_0 \\ \xi \end{pmatrix} \), \( \tilde{\mathbf{E}} = \begin{pmatrix} \eta_{00} & \eta^T \\ \eta & E \end{pmatrix} \).

- Potential: \( V = \tilde{\mathbf{K}}^T \tilde{\xi} + \tilde{\mathbf{K}}^T \tilde{\mathbf{E}} \tilde{\mathbf{K}} \).

- Three classes of stationary points:
  - \( K_0 = \mathbf{K} = 0 \):
    - Trivial solution \( (\varphi_1 = \varphi_2 = 0) \)
    - Unbroken \( SU(2)_L \otimes U(1)_Y \)
  - \( K_0 > \mathbf{K} \):
    - Solve \( \nabla_{\tilde{\mathbf{K}}} V = 0 \)
    - Fully broken \( SU(2)_L \otimes U(1)_Y \)
  - \( K_0 = \mathbf{K} > 0 \):
    - Solve \( \nabla_{\tilde{\mathbf{K}}_u} [V - u(K_0^2 - \mathbf{K}^2)] = 0 \)
    - \( SU(2)_L \otimes U(1)_Y \) broken to \( U(1)_{em} \)
    - None
    - Partial EWSB
    - Full EWSB

Solve
\[
\nabla_{\tilde{\mathbf{K}}_u} [V - u(K_0^2 - \mathbf{K}^2)] = 0
\]
\( SU(2)_L \otimes U(1)_Y \) broken to \( U(1)_{em} \)
Criteria for Symmetry Breaking

stationary points considerations easily give:

\[ V = \frac{1}{2} \tilde{K}^T \tilde{\xi} = -\tilde{K}^T \tilde{E} \tilde{K} \text{ at stationary points} \]

\[ \Rightarrow V < 0 \text{ for all non-trivial stationary points} \]

mutually exclusive possibilities for local minima:

- one or multiple min. with required EWSB \((K_0 = |K|)\)
- one charge breaking minimum \((K_0 > |K|)\)
- (degenerate set of solutions \((K_0 \geq |K|)\))
- trivial minimum \((\tilde{K} = 0)\)
- nontrivial minimum \(\Leftrightarrow \xi_0 < |\xi|\),

**Theorem**

Global minimum with spont. symmetry breaking \(SU(2)_L \otimes U(1)_Y \rightarrow U(1)_{em}\)

is given and guaranteed by stat. pnt. of type \(K_0 = |K| > 0\)
with largest Lagrange multiplier \(u_0 > 0\).
Potential after Symmetry Breaking

- **vacuum**

\[
\langle \varphi_1 \rangle \equiv \begin{pmatrix} 0 \\ v_1^0 \end{pmatrix}, \quad \langle \varphi_2 \rangle \equiv \begin{pmatrix} 0 \\ v_2^0 e^{i\xi} \end{pmatrix}
\]

must lie on the “forward light cone”

- **change of base** with \( \tan \beta = v_2^0 / v_1^0 \), unitary gauge:

\[
\langle \varphi'_1 \rangle \equiv \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} v_0^2 \end{pmatrix}, \quad \langle \varphi'_2 \rangle \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\( \Leftrightarrow \tilde{K}'^T = \begin{pmatrix} v_0^2 / 2 & 0 & 0 & v_0^2 / 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \)

separates Goldstone modes

- **charged mass squared**

\[
m_{H^\pm}^2 = 2u_0 v_0^2
\]
Example: THDM of Gunion et al.

Potential

- Higgs potential:

\[
V = \lambda_1 (\varphi_1^\dagger \varphi_1 - v_1^2)^2 + \lambda_2 (\varphi_2^\dagger \varphi_2 - v_2^2)^2 + \lambda_3 (\varphi_1^\dagger \varphi_1 - v_1^2 + \varphi_2^\dagger \varphi_2 - v_2^2)^2 \\
+ \lambda_4 ((\varphi_1^\dagger \varphi_1)(\varphi_2^\dagger \varphi_2) - (\varphi_1^\dagger \varphi_2)(\varphi_2^\dagger \varphi_1)) \\
+ \lambda_5 (\text{Re}(\varphi_1^\dagger \varphi_2) - v_1 v_2 \cos \xi)^2 + \lambda_6 (\text{Im}(\varphi_1^\dagger \varphi_2) - v_1 v_2 \sin \xi)^2 \\
+ \lambda_7 (\text{Re}(\varphi_1^\dagger \varphi_2) - v_1 v_2 \cos \xi)(\text{Im}(\varphi_1^\dagger \varphi_2) - v_1 v_2 \sin \xi)
\]

for all \( \lambda_i > 0 \) global minimum obvious:

\[
\langle \varphi_1 \rangle = \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \quad \langle \varphi_2 \rangle = \begin{pmatrix} 0 \\ v_2 e^{i\xi} \end{pmatrix}
\]

- stability conditions:

\[
\lambda_1 + \lambda_3 > 0, \quad \lambda_2 + \lambda_3 > 0, \quad \lambda_4, \kappa > -2\lambda_3 - 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}
\]

where \( \kappa := \frac{1}{2}(\lambda_5 + \lambda_6 - \sqrt{(\lambda_5 - \lambda_6)^2 + \lambda_7^2}) \).
Example: THDM of Gunion et al.

Figure: $V$ (shifted $V(\tilde{K} = 0)$) at all stationary points with $K_0 = |K| > 0$ for

$\lambda_3 = 0.1, \lambda_4 = 0.2, \lambda_5 = \lambda_6 = 0.4, v_1 = 30 \text{ GeV}, v_2 = 171 \text{ GeV}, \lambda_7 = 0, \xi = 0$. 

Stationary Points

$V(\tilde{K})$ (shifted $V(\tilde{K} = 0)$) at all stationary points with $K_0 = |K| > 0$ for $\lambda_3 = 0.1, \lambda_4 = 0.2, \lambda_5 = \lambda_6 = 0.4, v_1 = 30 \text{ GeV}, v_2 = 171 \text{ GeV}, \lambda_7 = 0, \xi = 0$. 

Yithsbey Giraldo; Universidad de Antioquia, Universidad de Nariño  
Stability and Symmetry Breaking in the THDM  
Seminario – p. 17
Summary

General THDM Higgs potential:

- orbit variables simplify access to structure:
  - no gauge d.o.f.
  - reduction of powers

- results for stability and for symmetry breaking
  - general conditions via univ. function
    (explicit solutions for stat. pnts., arbitrary basis)
  - structural statements:
    existence of minima, . . .

- specific application: clarifications for THDM of Gunion et al.
  - explicite general stability conditions
  - global minimum implications