Five Texture Zeros for Dirac Neutrino Mass Matrices

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In this work, we propose new five texture zeros for the mass matrices in the lepton sector in order to predict neutrino masses. In our approach, we extend beyond the Standard Model by assuming Dirac masses for the neutrinos, a feature which allows us to make a theoretical prediction for the lightest neutrino mass in the normal ordering. The textures that were analyzed have enough free parameters to adjust the $V_{\rm PMNS}$ mixing matrix including the CP-violating phase, the neutrino mass squared differences $\delta m^2_{21}, \delta m^2_{31}$, and the three charged lepton masses. In order to obtain reliable results, we used two different procedures that are based on the weak basis transformation, a well known technique to analyze textures and their implications for flavor physics. The first method was based on a least-squares analysis to theoretically fit the lepton masses and the mixing parameters to their corresponding experimental values; for this case, the best fit obtained for the lightest neutrino mass was $(3.9\pm^{0.6}_{0.8}) \times 10^{-3}$ eV. The second approach was algebraic, resulting in a lightest neutrino mass consistent with the experimental values and the restrictions arising from the five texture zeros of the mass matrices, and the lightest neutrino mass obtained was $(3.5 \pm 0.9) \times 10^{-3}$ eV.

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I. INTRODUCTION

The Standard Model (SM) of the strong and electroweak interactions [1] based on the local gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$, with the unbroken $SU(3)_c$ sector being confined and the electroweak sector $SU(2)_L \otimes U(1)_Y$ being spontaneously broken by one scalar complex Higgs doublet down to $U(1)_{em}$, has been very successful in explaining several experimental issues. Some of the unanswered questions of the model are the explanation of the total number of particle families present in nature, the hierarchy of the charged fermion mass spectrum, the quark and lepton mixing angles, and the origin of CP violation. Other mayor issues are the existence of dark energy and dark matter and the small magnitudes of the neutrino masses and their oscillations.

The extension of the SM with three right handed neutrinos (SMRHN) provides several useful features: a) it allows the introduction of nine additional complex Dirac mass terms in the neutral lepton sector [2]; b) it permits the implementation of the seesaw mechanism [3–9], and c) it is conducive for performing an analysis with Hermitian mass matrices in the lepton sector of the model (see appendix A). Although the first two features have been widely explored in the literature, the last one has rarely been considered. One of the purpose of this study is to analyze the mathematical and numerical consequences of the third feature mentioned above.

Texture zeros in the mass matrices of quarks and leptons have been an outstanding working hypothesis that provide the relationships between the mixing angles and the mass values in each sector. For the analysis of the idea dates back to the pionering work of H. Fritzsch [10–12], passing through the seminal work of Ramond, Robert and Ross [13, 14] with a complete and exhaustive analysis of the five texture zeros case, including the CP violation phenomena, reported in Ref. [15].

For the analysis of neutrinos, the situation is more complex because three different scenarios must be considered: left handed Majorana neutrinos, type I seesaw, and pure Dirac neutrinos. In the case where the SM neutrinos are massless, the simplest way to provide them with masses is to introduce a scalar $SU(2)_L$ triplet that develops vacuum expectation value (VEV), implying the existence of three Majorana left-handed fields and a Majoron which could be difficult to see. An exhaustive five texture zero analysis related with this 3×3 left-handed neutrino mass matrix has been presented in Ref. [16], supplemented by a more recent work in Ref. [17].

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When the SM is extended by including three right-handed neutrinos, the model can be used to explain neutrino masses via the type I seesaw mechanism. Then, the effective mass matrix of the light Majorana neutrinos is given by $M \sim M_D M_R^{-1} M_D^T$ where M_D and M_R are 3×3 mass matrices for the Dirac and Right-Handed (RH) Majorana neutrinos, respectively. For this case, the texture analysis becomes more complicated, even though this texture can be simply imposed on the 3×3 light neutrino mass matrix directly. However, in the context of the type I seesaw mechanism, since the light mass matrix is a product of the Dirac mass matrix and the RH neutrino mass matrix, it is natural to consider texture zeros in both matrices, the Dirac and the RH neutrino mass matrices. The analysis for this case has been presented in full detail in Ref. [18]. The case of pure Dirac neutrino masses is the one we will address in this study, based on a previous study that has already been reported in Ref. [17] and in which, Hermitian mass matrices were also considered.

The texture of the Dirac neutrinos mass matrix has been the subject of several recent studies [17, 19–30]; this, in part, has been motivated by the non-observation of the neutrinoless double beta decay [31]. As it is well known, existing experiments have not been able to determine if neutrinos are Majorana or Dirac particles; however, most works on these topics assume that neutrinos are Majorana, whereas the case for Dirac neutrino masses has not been studied in detail. In this study we assume that neutrinos are Dirac particles [2, 32, 33], which allows us to use the weak basis transformation, the polar decomposition theorem, and the more recent experimental data to make predictions based on a given texture for the lepton mass matrices. It is interesting to determine the consequences of a given texture on the prediction of the neutrino masses; indeed, for Majorana masses, there are several works predicting a mass of a few milli-electron volts for the lightest neutrino mass [34–37]; as we will see, this value of mass has the same order of magnitude as the results presented in this work.

There are theoretical motivations to assume Dirac masses for the neutrinos. Dirac neutrino masses are also useful to generate baryon asymmetry via leptogenesis [38], alternative approaches to the seesaw mechanism [39], and radiative neutrino masses [40–51], in addition to other phenomenological motivations [52, 53]. In models derived from string theory, the Majorana masses are suppressed by selection rules related to the underlying symmetries [54, 55]. In these models, Majorana masses can also be generated by active neutrinos via gravitational effects; however, these masses are very small, when compared with the small scales in neutrino physics [56, 57]. The Dirac neutrino electromagnetic properties can be tested easily due to the magnetic dipole moment that is different from zero at the quantum level [58]. The same is not true for Majorana neutrinos, for which their electromagnetic properties are not easily because they are their own antiparticles. In this way, theoretical models with Dirac neutrinos are a motivation to refine the electromagnetic constraints on the neutrino properties.

In the SMRHN and forbidding the bare Majorana masses of the right-handed neutral states (obtained just by assuming lepton number conservation), the lepton mass terms after the spontaneous breaking of the local symmetry are given by

$$-\mathcal{L}_D = \bar{\nu}_L M_n \nu_R + \bar{\ell}_L M_\ell \ell_R + \text{h.c}, \tag{1}$$

where $\nu_{L,R} = (\nu_e, \nu_\mu, \nu_\tau)^T_{L,R}$ and $\ell_{L,R} = (e, \mu, \tau)^T_{L,R}$ (the superscript T stands for transpose). M_n and M_ℓ are the 3×3 complex mass matrices for the neutral and charged lepton sector, respectively. In the most general case, they contain 36 free parameters. In the context of the SMRHN, such a large number of parameters can be drastically cut by making use of the polar theorem of matrix algebra, by which, one can always decompose a general complex matrix as the product of a Hermitian and a unitary matrix; in this way, the unitary matrix can be absorbed in a redefinition of the right handed lepton components, and this immediately reduces the number of free parameters from 36 to 18 (the other eighteen parameters can be hidden in the right-handed lepton components in the context of the SM and some of its extensions, but not for the left-right symmetric extensions). Hence, as far as this model is concerned, we may treat without loss of generality M_n and M_ℓ as two Hermitian mass matrices, with 18 real parameters in total, out of which six are phases. Since five of those phases can be absorbed in a redefinition of the lepton fields [59, 60], the total number of free parameters we may play with in M_n and M_ℓ are 12 real parameters and one phase; the last parameter can be used to explain the CP violation phenomena.

As mentioned above, we extend the SM to include the right-handed neutrinos and consider them to be Dirac type particles. Thus, the mechanism implemented in the quark sector can be transferred with few modifications directly to the lepton sector. Then, in the context of the SMRHN, it is always possible to implement the so-called weak basis transformation (WBT) [61–63]. A WBT is a unitary transformation acting simultaneously on the neutral and charged lepton fermion mass matrices. That is,

$$M_n \to M_n^R = U M_n U^\dagger, \qquad M_\ell \to M_\ell^R = U M_\ell U^\dagger, \tag{2}$$

where U is an arbitrary unitary matrix. We say then that the two representations (M_n, M_ℓ) and (M_n^R, M_ℓ^R) are equivalent in the sense that they are related to the same Pontecorvo-Maki-Nakagawa-Sakata matrix (PMNS matrix).

This kind of transformation plays an important role in the study of the so-called flavor problem.

In references [15, 61, 62], it was shown that for Hermitian mass matrices, it is always possible to perform a WBT in such a way that it is always possible to arrive at with two mass matrices with three texture zeros that do not have any physical implication. With these three non-physical texture zeros, the number of free parameters in M_n and M_ℓ reduces from twelve to nine real parameters and one phase, just enough to fit the physical values for the six Dirac lepton masses, the three mixing angles, and the CP violation phase. However, for the case of neutrinos, their masses are not known, and only the square mass diferences are experimentally available, i.e., $\delta m^2_{21} = m^2_2 - m^2_1$ and $\delta m^2_{31} = m^2_3 - m^2_1$. As explained in Appendix A, for general 3×3 complex mass matrices in the lepton sector, without losing generality and after all the theoretical considerations, we are left with 9 free parameters and 8 experimental restrictions. To ensure an identical number of variables and constraints, it is necessary to include an additional texture-zero (totalling to four-texture zeros) to determine all the parameters, including the lightest neutrino mass (For a review on these topics, see [12]). Since in this case we have 8 experimental values and 8 free parameters, a solution is guaranteed. By imposing a second texture-zero (i.e., five-texture zeros), highly-non trivial mass matrices are required since there are more parameters than experimental restrictions. Given the small number of ansatz to fit the data, we can consider our solutions as a very representative set of the mass matrices with five texture zeros. In the next step, a fifth texture zero should provide us with one physical prediction, and a possible sixth texture zero will provide us with two predictions (On counting the degrees of freedom, we find that the maximum number of such texture zeros consistent with the absence of a zero mass eigenvalue and a nondegenerate mass spectrum in the lepton sector is just six, with three in the neutral sector and three in the charged sector).

In the analysis that follows, we use the SM ingredients that include one single complex Higgs doublet, enlarged with three right-handed neutrinos (one for each family) in order to provide the neutral sector with Dirac masses, and an unknown symmetry that is able to produce five texture zeros in the lepton sector. Two cases are going to be considered: one with three texture zeros in the neutral lepton sector and two in the charged one, and a second case with two texture zeros in the neutral lepton sector and three in the charged one. Two different analyses are implemented, one for each situation. The analytical and numerical results are presented, taking special care to accommodate the latest experimental data available [64], including the CP violation phenomena.

II. FIVE TEXTURE ZEROS: FIRST CASE.

In the context of the SMRHN with lepton number conservation, in the weak basis, and after breaking the local gauge symmetry, the Lagrangian mass term for the lepton sector is given by

$$-\mathcal{L}_D = \bar{\nu}'_L M'_n \nu'_R + \bar{\nu}'_R M'^{\dagger}_n \nu'_L + \bar{\ell}'_L M'_\ell \ell'_R + \bar{\ell}'_R M'^{\dagger}_\ell \ell'_L, \tag{3}$$

where M'_n and M'_ℓ are the neutrino and charged lepton mass matrices respectively (primed fields and matrices refer to the weak basis).

Let us now assume that for a given symmetry, the Hermitian mass matrices M'_n and M'_ℓ present the following textures

$$M'_n = \begin{pmatrix} c_n & a_n & 0 \\ a^*_n & 0 & b_n \\ 0 & b^*_n & 0 \end{pmatrix}, \qquad M'_\ell = \begin{pmatrix} 0 & a_\ell & 0 \\ a^*_\ell & d_\ell & b_\ell \\ 0 & b^*_\ell & c_\ell \end{pmatrix}. \tag{4}$$

In what follows, we present an analysis of the consequences of this particular pattern with three texture zeros in the neutral sector and two in the charged one.

The first step is to remove the phases; this can be done by the following unitary transformation:

$$M'_{n,\ell} = \lambda^{\dagger}_{n,\ell} M_{n,\ell} \lambda_{n,\ell}, \tag{5}$$

which is achieved by using the diagonal matrices $\lambda_n = (1, e^{i\alpha_{n_1}}, e^{i\alpha_{n_1}+i\alpha_{n_2}})$ and $\lambda_\ell = (1, e^{i\alpha_{\ell_1}}, e^{i\alpha_{\ell_1}+i\alpha_{\ell_2}})$, respectively, and $M_{n,\ell}$ are the matrices whose components are the absolute values of the corresponding entries in $M'_{n,\ell}$ (i.e., $(M_{n,\ell})_{i,j} = |(M'_{n,\ell})_{i,j}|$). If we rotate these matrices by using the orthogonal transformation $R_{n,\ell}$ ($R^T_{n,\ell} R_{n,\ell} = \mathbb{1}$) to the mass eigenstate space (the physical basis), we obtain

$$M'_{n,\ell} = \lambda^{\dagger}_{n,\ell} R^T_{n,\ell} \begin{pmatrix} m_{1,e} & 0 & 0 \\ 0 & -m_{2,\mu} & 0 \\ 0 & 0 & m_{3,\tau} \end{pmatrix} R_{n,\ell} \lambda_{n,\ell} \equiv U_{n,\ell} M^{\mathrm{diag}}_{n,\ell} U^{\dagger}_{n,\ell}, \tag{6}$$

where at least one negative eigenvalue is needed in order to generate a texture-zero in the diagonal [62, 65, 66]. Here m_1, m_2 and m_3 are the masses of the electron, muon and tau neutrinos respectively, with the masses of the charged leptons (in MeV) given by: $m_e = 0.5109989461 \pm 0.0000000031, m_\mu = 105.6583745 \pm 0.0000024$ and $m_\tau = 1776.86 \pm 0.12$, which correspond to the electron, muon and tau masses, respectively [64]. After rotating to the mass eigenstates, the eigenvalues can be positive, negative or zero. In these expressions, M_n^{diag} and M_ℓ^{diag} are the diagonal mass matrices for the neutrino and charged lepton sectors, respectively. In accordance with the standard notation, we use $U_n \equiv (R_n \lambda_n)^\dagger$ and $U_\ell \equiv (R_\ell \lambda_\ell)^\dagger$, that are two unitary matrices used to rotate from the weak basis to the physical basis. From Eqs. (3) and (4), we obtain the relation between the states in the mass basis $\nu_{L,R}$, $\ell_{L,R}$, and the corresponding states in the interaction basis $\nu'_{L,R}$, $\ell'_{L,R}$:

$$\nu'_{L,R} = U_n \nu_{L,R}, \qquad \ell'_{L,R} = U_\ell \ell_{L,R}. \tag{7}$$

Replacing these expressions in the lepton sector of the weak current, we obtain

$$\mathcal{L}_{W^-} = -\frac{g}{\sqrt{2}} W^- \bar{\ell}'_L \gamma^\mu \nu'_L + \text{h.c} = -\frac{g}{\sqrt{2}} W^- \bar{\ell}_L \gamma^\mu U_\ell^\dagger U_n \nu_L + \text{h.c}, \tag{8}$$

in such a way that the PMNS matrix is given by

$$V_{\text{PMNS}} = U_\ell^\dagger U_n = R_\ell \, \Phi \, R_n^T, \tag{9}$$

where $\Phi = \lambda_\ell \lambda_n^\dagger$ is a diagonal phase matrix. For the neutrino mass matrix, normal ordering is assumed [67], i.e.,: $m_3 > m_2 > m_1$, where: $m_2^2 = m_1^2 + \delta m_{21}^2$, and $m_3^2 = m_1^2 + \delta m_{31}^2$, with $\delta m_{21}^2, \delta m_{31}^2 > 0$ [68].

By imposing the invariance of the trace and the determinant of the mass matrices ($\text{tr}[M'_{n,\ell}] = \text{tr}[M_{n,\ell}^{\text{diag}}]$, $\text{tr}\left[\left(M'_{n,\ell}\right)^2\right] = \text{tr}\left[\left(M_{n,\ell}^{\text{diag}}\right)^2\right]$, and $\text{Det}[M'_{n,\ell}] = \text{Det}[M_{n,\ell}^{\text{diag}}]$), the following relations are obtained for this particular texture:

$$\begin{aligned}
c_n &= m_1 - m_2 + m_3, & d_\ell &= m_e - m_\mu + m_\tau - c_\ell, \\
|a_n| &= \sqrt{\frac{(m_1 - m_2)(m_1 + m_3)(m_2 - m_3)}{m_1 - m_2 + m_3}}, & |b_\ell| &= \sqrt{\frac{(c_\ell - m_e)(c_\ell + m_\mu)(m_\tau - c_\ell)}{c_\ell}}, \\
|b_n| &= \sqrt{\frac{m_1 \, m_2 \, m_3}{m_1 - m_2 + m_3}}, & |a_\ell| &= \sqrt{\frac{m_e \, m_\mu \, m_\tau}{c_\ell}}.
\end{aligned}$$

From the previous identifications, it is possible to obtain an explicit form for the mass matrices of leptons that allows us to obtain, through diagonalization of M_n and M_ℓ, the orthogonal matrices in Eq. (9),

$$R_n = \begin{pmatrix}
-\sqrt{\frac{m_1(m_2-m_1)(m_1+m_3)}{(m_1+m_2)(m_3-m_1)(m_1-m_2+m_3)}} & \sqrt{\frac{m_1(m_3-m_2)}{(m_1+m_2)(m_3-m_1)}} & \sqrt{\frac{m_2 m_3(m_3-m_2)}{(m_1+m_2)(m_3-m_1)(m_1-m_2+m_3)}} \\
\sqrt{\frac{m_2(m_1-m_2)(m_2-m_3)}{(m_1+m_2)(m_2+m_3)(m_1-m_2+m_3)}} & -\sqrt{\frac{m_2(m_1+m_3)}{(m_1+m_2)(m_2+m_3)}} & \sqrt{\frac{m_1 m_3(m_1+m_3)}{(m_1+m_2)(m_2+m_3)(m_1-m_2+m_3)}} \\
\sqrt{\frac{m_3(m_1+m_3)(m_3-m_2)}{(m_3-m_1)(m_2+m_3)(m_1-m_2+m_3)}} & \sqrt{\frac{m_3(m_2-m_1)}{(m_2+m_3)(m_3-m_1)}} & \sqrt{\frac{m_1 m_2(m_2-m_1)}{(m_3-m_1)(m_2+m_3)(m_1-m_2+m_3)}}
\end{pmatrix}, \tag{10}$$

$$R_\ell = \begin{pmatrix}
-\sqrt{\frac{m_\mu m_\tau(c_\ell-m_e)}{c_\ell(me+m_\mu)(m_\tau-m_e)}} & -\sqrt{\frac{m_e(c_\ell-m_e)}{(m_e+m_\mu)(m_\tau-m_e)}} & \sqrt{\frac{m_e(c_\ell+m_\mu)(c_\ell-m_\tau)}{c_\ell(m_e+m_\mu)(m_e-m_\tau)}} \\
\sqrt{\frac{m_e m_\tau(c_\ell+m_\mu)}{c_\ell(m_e+m_\mu)(m_\mu+m_\tau)}} & -\sqrt{\frac{m_\mu(c_\ell+m_\mu)}{(m_e+m_\mu)(m_\mu+m_\tau)}} & \sqrt{\frac{m_\mu(m_e-c_\ell)(c_\ell-m_\tau)}{c_\ell(m_e+m_\mu)(m_\mu+m_\tau)}} \\
\sqrt{\frac{m_e m_\mu(c_\ell-m_\tau)}{c_\ell(m_e-m_\tau)(m_\mu+m_\tau)}} & \sqrt{\frac{m_\tau(m_\tau-c_\ell)}{(m_\tau-m_e)(m_\mu+m_\tau)}} & \sqrt{\frac{m_\tau(c_\ell-m_e)(c_\ell+m_\mu)}{c_\ell(m_\tau-m_e)(m_\mu+m_\tau)}}
\end{pmatrix}. \tag{11}$$

From R_n and R_ℓ, the PMNS mixing matrix defined in Eq (9) can be constructed. This is a matrix, which besides the CP violating phase, is a function of a single mathematical parameter c_ℓ. In this way, the three mixing angles in V_{PMNS} are expressed as functions of the lepton masses, c_ℓ, and in practice, also of the phases, with two physical predictions according to the parameter counting analysis presented in Appendix A. The entries in R_n and R_ℓ are real values because of the normal hierarchy assumed in the neutral lepton sector and as far as c_ℓ is in the interval $m_e < c_\ell < m_\tau$. We then have the freedom to use c_ℓ and m_1 as free parameters fixed by a statistical analysis.

A. Least squares analysis

From Eq. (9), we know that $V_{\text{PMNS}} = R_\ell \, \Phi \, R_n^T$, with Φ as the following diagonal matrix:

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_1} & 0 \\ 0 & 0 & e^{i\phi_2} \end{pmatrix},$$

where ϕ_1 and ϕ_2 are free parameters.

Then, the former analysis implies that the PMNS matrix is a function of the free parameters $(m_1, c_\ell, \phi_1, \phi_2)$, wherein we have chosen m_1 as the lightest neutrino mass. After a numerical adjustment by means of the χ^2 analysis, it is found that for our particular choice of the five-zero texture lepton mass matrices in Eq. (4), the normal hierarchy is favored. Neglecting correlations, the χ^2 function is given by

$$\chi^2 = P_J^2 + \sum_{i,j=1,2,3} P_{ij}^2,$$

where the pulls are

$$P_{ij} = \frac{U_{ij} - \bar{U}_{ij}}{\delta U_{ij}}$$

where $U_{ij} = |(V_{\text{PMNS}})_{ij}|$ is the absolute value of the components from the product of the diagonalization matrices (9). The absolute values $\bar{U}_{ij}$ correspond to the global averages for the components of the PMNS matrix and δU_{ij} correspond to 1σ errors. P_J is the pull of the Jarlskog invariant, which in the standard parameterization, is given by $\bar{J} = c_{12}c_{23}c_{13}^2 s_{12}s_{23}s_{13}\sin\delta = -0.0270054$ and the corresponding 1σ uncertainty is $\delta J = 0.0106304$, for normal ordering [64]. The theoretical prediction is given by $J = \text{Im}\,(U_{\mu 3}U_{\tau 3}^* U_{\mu 2} U_{\tau 2}^*)$, where in this expression U stands for the PMNS mixing matrix. The upper bound $m_1 + m_2 + m_3 < 0.17\,\text{eV}$ [64, 69–71] , is also imposed. Using the data from [68] [1], the fit results are shown in the following tables:

m_1 (eV)	c_ℓ (eV)	ϕ_1 (rad)	ϕ_2 (rad)	χ^2_{min}
$0.00395 \pm^{0.00062}_{0.00078}$	523176.	0.0190664	1.56122	12.4204

TABLE I: Best fit free parameters and minimum χ^2 function.

P_{11}	P_{12}	P_{13}	P_{21}	P_{22}	P_{23}	P_{31}	P_{32}	P_{33}	P_J
0.428531	-0.385085	0.0430767	-0.205321	-1.2577	1.91336	0.290472	1.25036	-2.2701	0.0228083

TABLE II: $P_{i,j}$ is the pull of the i,j component of the PMNS matrix and P_J is the pull of the Jarlskog invariant in the χ^2 analysis. The minimum of the χ^2 function is 12.4204 for ten observables and four parameters $(m_1, c_\ell, \phi_1, \phi_2)$. The fit goodness is $\chi^2/\text{d.o.f} = 2.07$ which is a relatively high value due to P_{33} and P_{23} pulls which have a deviation around 2σ respect to their experimental values, despite this result it is still an acceptable fit.

In our χ^2 analysis, the pseudo observables are the absolute values of the PMNS matrix components and the Jarlskog invariant, with pulls $P_{i,j}$ and P_J, respectively. We do not consider the correlations between them [2].

Even though a value for the fit goodness $\chi^2/\text{d.o.f} \sim 2.07$ is not optimal, the result is acceptable. We can see that the main source of discrepancy is related to the $(V_{\text{PMNS}})_{23}$ and $(V_{\text{PMNS}})_{33}$ components, which deviate from their experimental values by 2σ. It must be emphasized here that a lightest neutrino mass equal to zero and the inverse ordering of the neutrino masses are not favored by this texture (the same is true for the equivalent textures via WBT).

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[1] NuFIT collaboration (http://www.nu-fit.org/?q=node/211)(with SK atmospheric data).

[2] The collaborations report correlation effects between observables, in our case, the components of the PMNS matrix are the result of a global fit. However, in phenomenology, it is a common practice to use pseudo observables.

It is possible that for another non-equivalent five-zero texture, a realization of the inverse ordering may be needed, and this subject requires a more dedicated study.

III. FIVE TEXTURE ZEROS: SECOND CASE.

Let us now assume that in the context of the SMRHN with the neutrinos being only Dirac type particles, there is a symmetry that produces the Hermitian mass matrices M_n' and M_ℓ' with the following textures:

$$M_n' = \begin{pmatrix} 0 & C_n & 0 \\ C_n^* & E_n & B_n \\ 0 & B_n^* & A_n \end{pmatrix}, \qquad M_\ell' = \begin{pmatrix} 0 & C_\ell & 0 \\ C_\ell^* & 0 & B_\ell \\ 0 & B_\ell^* & A_\ell \end{pmatrix}. \tag{12}$$

Let us analyze the consequences of this new pattern with three texture zeros in the charged sector and two in the neutral one. Without losing generality, it is possible to eliminate the phases of the M_ℓ' matrix by means of a WBT, so that the CP violation phase only appears in the neutrino mass matrix. The algebra shows that it is possible to diagonalize the charged lepton sector $M_\ell = U_\ell D_\ell U_\ell^\dagger$ (which as in the previous section we define $(M_\ell)_{i,j} = |(M_\ell')_{i,j}|$), where D_ℓ =Diag.$(m_e, -m_\mu, m_\tau)$, in order to take full advantage of using the following unitary matrix [72]:

$$U_\ell = \begin{pmatrix} e^{i\theta_1}\sqrt{\frac{m_\mu m_\tau (A_\ell - m_e)}{A_\ell (m_\mu + m_e)(m_\tau - m_e)}} & -e^{i\theta_2}\sqrt{\frac{m_e m_\tau (m_\mu + A_l)}{A_\ell (m_\mu + m_e)(m_\tau + m_\mu)}} & \sqrt{\frac{-m_e m_\mu (A_\ell - m_\tau)}{A_\ell (m_\tau - m_e)(m_\tau + m_\mu)}} \\ e^{i\theta_1}\sqrt{\frac{m_e (m_e - A_\ell)}{(-m_\mu - m_e)(m_\tau - m_e)}} & e^{i\theta_2}\sqrt{\frac{m_\mu (A_\ell + m_\mu)}{(m_\mu + m_e)(m_\tau + m_\mu)}} & \sqrt{\frac{m_\tau (m_\tau - A_\ell)}{(m_\tau - m_e)(m_\tau + m_\mu)}} \\ -e^{i\theta_1}\sqrt{\frac{m_e (A_l + m_\mu)(A_\ell - m_\tau)}{A_\ell (-m_\mu - m_e)(m_\tau - m_e)}} & -e^{i\theta_2}\sqrt{\frac{m_\mu (A_\ell - m_e)(m_\tau - A_\ell)}{A_\ell (m_\mu + m_e)(m_\tau + m_\mu)}} & \sqrt{\frac{m_\tau (A_l - m_e)(A_\ell + m_\mu)}{A_\ell (m_\tau - m_e)(m_\tau + m_\mu)}} \end{pmatrix}, \tag{13}$$

where θ_1 and θ_2 are arbitrary phases and $A_\ell = m_e - m_\mu + m_\tau$. Even though the phases θ_1 and θ_2 in the rotation matrix of U_l are not CP phases (they can be absorbed in the fields), these phases are quite useful to match our theoretical expression for the PMNS matrix using the standard convention [73]. To obtain the three texture zeros in the lepton mass matrix, the following relations are also necessary:

$$|B_\ell| = \sqrt{\frac{(A_\ell - m_e)(A_\ell + m_\mu)(m_\tau - A_\ell)}{A_\ell}} \qquad \text{and} \qquad |C_\ell| = \sqrt{\frac{m_e\, m_\mu\, m_\tau}{A_\ell}}.$$

For the neutrino sector, we are subject to the condition $U_\ell^\dagger U_n = V_{\text{PMNS}}$, and necessarily, the diagonalizing matrix must be given by $U_n = U_\ell V_{\text{PMNS}}$, so that the relation between the mass matrix in the weak basis and the diagonal matrix D_n in the mass space is

$$M_n' = \begin{pmatrix} 0 & C_n & 0 \\ C_n^* & E_n & B_n \\ 0 & B_n^* & A_n \end{pmatrix} = U_\ell (V_{\text{PMNS}}) D_n (V_{\text{PMNS}})^\dagger U_\ell^\dagger \equiv U_n D_n U_n^\dagger. \tag{14}$$

For this second case the only free parameters are: m_1 from the diagonal matrix D_n=Diag.$(m_1, -m_2, m_3)$, and θ_1 and θ_2 from U_ℓ. This result is important since we can interpret the neutrino masses as predictions associated with the texture of the mass matrices. From these expressions, we can obtain useful relations by identifying U_ℓ with the WBT U in Eq. (2) [65].

A. Numerical results

For the second case, when solving numerically to obtain the textures for the neutrino mass matrix in the normal hierarchy, we obtain

$$\begin{aligned} m_1 &= (0.00354 \pm 0.00088)\,\text{eV}, \\ m_2 &= (0.00930 \pm 0.00036)\,\text{eV}, \\ m_3 &= (0.05040 \pm 0.00030)\,\text{eV}. \end{aligned} \tag{15}$$

In our numerical analysis the main source of uncertainty arises from the CP violation phase, and this is understandable because in the lepton sector, this parameter has not been determined with good precision. The associated numerical entries for the lepton mass matrices with five texture zeros are (in eV): $A_n = 0.0251821$, $B_n = (-0.0122955 + 0.0244187i)$, $C_n = (0.00427236 + 0.00689527i)$, $E_n = 0.0194623$, $A_\ell = 1671.71 \times 10^6$, $|B_\ell| = 432.237 \times 10^6$, $|C_\ell| = 7.57544 \times 10^6$, and phases $\theta_1 = 0.154895$ and $\theta_2 = 2.01797$. The phases of B_ℓ and C_ℓ were absorbed in B_n and C_n by means of a redefinition, through a WBT, in a previous step.

By construction, the WBT formalism reproduces the mixing matrix, the mass of charged leptons, and the neutrino mass squared differences. As input parameters, we used the central values of the global fit reported by the Nu-FIT collaboration (with SK atmospheric data) [68]. When comparing with the method of least squares, in the WBT formalism, the numerical results do not deviate from the experimental values, as shown in Table III.

θ_{12} (°)	θ_{23} (°)	θ_{13} (°)	δ_{CP} (°)	δm^2_{21} (eV^2)	δm^2_{31} (eV^2)	m_e (MeV)	m_μ (MeV)	m_τ (MeV)
33.82	48.6	8.60	221	7.39×10^{-5}	2.528×10^{-3}	0.510999	105.658	1776.86

TABLE III: Output values in our analysis.

IV. CONCLUSIONS

In this analysis, we explore the consequences of extending the SM with three right-handed neutrinos that allow nine additional complex Dirac mass terms for the neutral lepton sector, excluding the possibility of having bare Majorana masses. In this extended model, it is not possible to determine the neutrino masses from first principles; however, as it is well known in the literature, by imposing discrete or continuous symmetries or, in an equivalent way, a texture for the lepton mass matrices, it is possible to determine the neutrino masses. This is not a trivial exercise since the number of texture-zeros and the corresponding free parameters must be adjusted in order to obtain consistent physical results and predictions. Under these conditions, an ansatz for the lepton mass matrices emerges from the quark-lepton similarity, allowing us to extend the analysis of the mass matrices from the quark sector to the lepton sector, which is a natural and important question. This allowed us to consider, without losing generality, the mass matrices of the lepton sector as Hermitian in such a way that it is possible to apply the WBT formalism without any restriction. Taking advantage of the large number of techniques developed in the quark sector, the texture-zeros facilitate the analysis for obtaining the lepton masses and the PMNS mixing matrix. The texture-zeros in the lepton mass matrices can be derived from additional hidden symmetries that do not allow certain inputs into the Yukawa Lagrangian; however, this is not the purpose of the present work and we leave this exploration for future studies.

Two different five texture zeros for the Hermitian lepton mass matrices were considered, one with three zeros in the neutral sector and two in the charged sector, and the other one with two texture zeros in the neutral sector and three in the charged one. In order to obtain reliable results, we used two different approaches, assuming for both a normal ordering for the neutrino physical masses. By counting the degrees of freedom in the lepton sector, and after making use of the polar theorem of matrix algebra and the consequences of the WBT, we have concluded that with five texture zeros in the Hermitian lepton mass matrices, only one prediction can be achieved. For the first case, we started by considering the five independent texture structures presented for quarks in Ref. [13, 14], and modified them until we determined the optimum structure that has been reported in this study. To the best of our knowledge, the texture analyzed in our study has not been considered in the literature so far. For the second case, we used the second form given in Ref [13, 14]. In both cases, the mass of the lightest neutrino can be considered as a prediction of the models studied.

The first analysis, based on a least squares approach, was used to adjust the lepton masses and the mixing parameters to their corresponding experimental values. It was implemented for the texture with three zeros in the neutral sector and two in the charged one. In this approach, the fit of the mixing parameters to the reported values in the literature is below two sigmas, with an acceptable goodness of fit. The best fit for the lightest neutrino mass in this case was $m_1 \approx (3.9\pm^{0.6}_{0.8}) \times 10^{-3}$ eV, which is similar to reported values based on other assumptions [34–36].

The second analysis was a purely algebraic and numerical study, based on the WBT approach. It was implemented for the texture with three zeros in the charged lepton sector and two in the neutral one. The prediction for the lightest neutrino mass was $(3.5 \pm 0.9) \times 10^{-3}$ eV, which is in agreement with the previous result. It must be noted that in this case, the numerical results do not deviate from the experimental results.

For the case of Majorana masses, there are several studies predicting a mass of a few milli-electron volts for the lightest neutrino [34–36, 74, 75], and these values are of the same order of magnitude as the results reported in this work. Most of these results are reported by Fritzsch et al., except the reference [75], where the problem is analyzed by assuming hierarchical neutrino masses and minimal flavor symmetry breaking. In this work, they claim that their analysis is valid for Dirac or Majorana masses. Subsequently, they reported a mass of a few meV for the lightest neutrino.

The two different five texture-zeros proposed in Eqs. (4) and (12) are not equivalent in the sense that there is no WBT relating them.

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Appendix A: Parameter counting

In this section, we will show that the lepton mass matrices can be considered Hermitian without any loss of generality. After spontaneous symmetry breaking of $SU(2) \otimes U(1)_X \to U(1)_Y$, the Yukawa Lagrangian of the lepton sector assumes the following form in the space of interactions:

$$-\mathcal{L}_D = \bar{\ell}'_L M'_\ell \ell'_R + \bar{\nu}'_L M'_n \nu'_R + \text{h.c}, \tag{A1}$$

where $\nu'_{L,R} = (\nu'_e, \nu'_\mu, \nu'_\tau)^T_{L,R}$ and $\ell'_{L,R} = (e', \mu', \tau')^T_{L,R}$ (the upper T stands for transpose). The most general mass matrices of the charged M'_ℓ and neutral M'_n sectors, contain 36 free parameters. The polar decomposition theorem [76, 77] states that any matrix T, real or complex, can be written as

$$T = HU,$$

where H is a positive operator (Hermitian operator with positive eigenvalues) and U is a unitary matrix. Therefore, we can write the mass matrices as follows:

$$M'_\ell = H'_\ell U'_\ell, \qquad M'_n = H'_n U'_n. \tag{A2}$$

Since the right-handed fermions are singlets under the SM group, the unitary matrix can be absorbed in these fields [3] in such a way that we can write the Lagrangian in terms of Hermitian mass matrices:

$$-\mathcal{L}_D = \bar{\ell}'_R H'_\ell \ell'_L + \bar{\nu}'_R H'_n \nu'_L + \text{h.c.}$$

H'_ℓ and H'_n are defined positive, however, negative eigenvalues are needed to have zeros in the diagonal [62], and this can be solved easily by redefining the right fields with a phase. The two mass matrices M_ℓ and M_n are arbitrary complex 3×3 matrices with 36 free parameters, and by limiting our analysis to Hermitian matrices, we halved this number. Of the remaining 18 free parameters (i.e., the number of off-diagonal elements in both matrices divided into 2), six are phases, five of them can be absorbed in a redefinition of the lepton fields [59, 60], from which, one CP violating phase remains.

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[3] There are cases where this process does not apply. For example, in left-right handed models where the right fields transform under SU(2), the unitary component cannot be absorbed.

With 12 real free parameters, we have to explain: three charged lepton masses, two squared mass differences for the neutral sector, and three mixing angles, totalizing 8 experimental restrictions. To make predictions, it is usual to set zero some entries of the mass matrices; however, as shown in the references [12, 61, 62], given any two 3×3 quark mass matrices M_ℓ, M_n there is always a WBT such that the new mass matrices M'_ℓ, M'_n have three texture-zeros, without implying a relation between the physical quantities. That is to say, of the 12 real parameters (which could be mass matrix elements) it is possible to make 3 of them equal to zero in such a way that finally we are left with 9 free parameters and 8 experimental restrictions.

To ensure an identical number of variables and constraints it is necessary an additional texture-zero (equivalent to 4 texture zeros) such that it is possible to solve for all the parameters in the mass matrices, including for the lightest neutrino mass. With two physical texture-zeros (i.e., five texture zeros in M_ℓ and M_n) the number of free parameters reduces to 7 which is precisely the number of real parameters in our mass matrices. In this case, the number of experimental restrictions exceeds the number of free parameters, the problem is over-constrained and not all textures are going to be consistent with the experimental values. The textures reported in this work can adjust the 8 physical quantities simultaneously with 7 real parameters representing a highly non-trivial result (For a review on these topics, see [12]).

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