Los complejos cúbicos CAT(0) en la robótica

(y en muchos otros lugares)

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ALTENCOA 2014
Pasto, Nariño
11 de agosto de 2014

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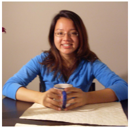
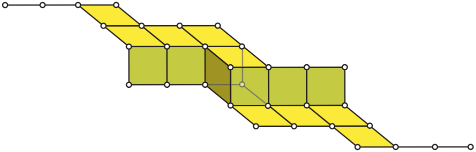
- Diego Cifuentes (Los Andes/MIT), Steven Collazos (SFSU)

Las dos cosas que quiero decir hoy:

1. Hay muchos complejos cúbicos CAT(0) “en la naturaleza".

2. Los complejos cúbicos CAT(0) tienen una estructura muy elegante y muy útil.



1. MOTIVATION.

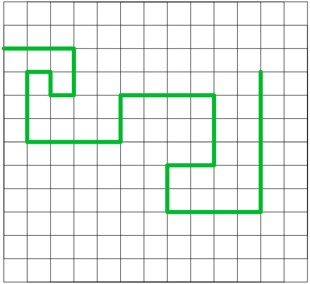
Moving robots.

A robotic snake can move:

1. the head or tail or 2. a joint

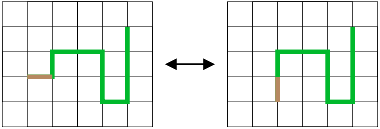
without self-intersecting.

Snake:

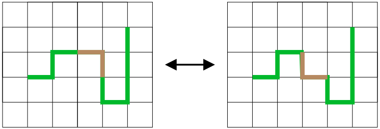


Moves:

1:



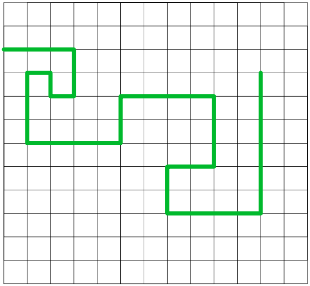
2:

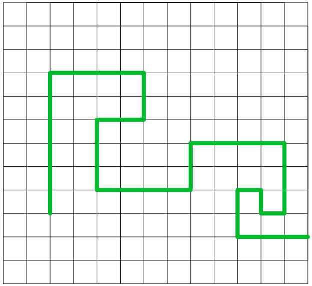


One motivation: moving robots.

How do we move this robotic snake (optimally) using these moves from one position to another one?

Position 1 $\rightarrow$ Position 2





Motivación: una pregunta más fácil.

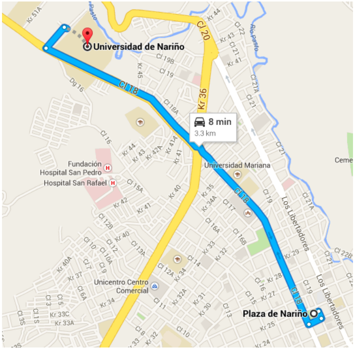
Cómo llego a la universidad?

Plaza de Nariño $\rightarrow$ Universidad de Nariño

Motivación: una pregunta más fácil.

¿Cómo llego a la universidad, **de manera óptima**?

¡Con un mapa! (Ojo: ¿Óptima en qué sentido?)

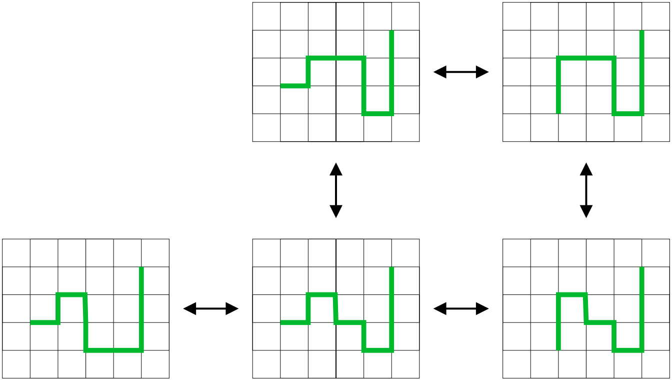


Hagamos lo mismo.
Constuyamos un mapa de las posibles posiciones del robot.

Motivation: back to moving robots.

Let's build a map of all possible positions of the robot.

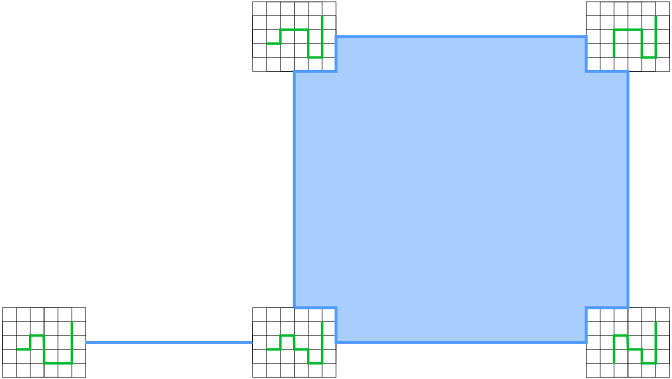
A small piece: (discrete model)



Motivation: moving robots.

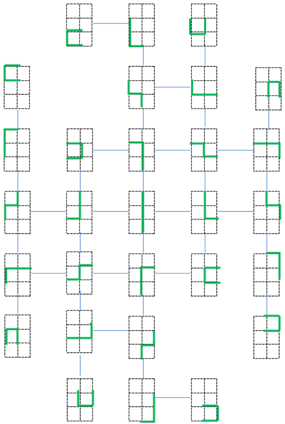
Let's build a map of all possible positions of the robot.

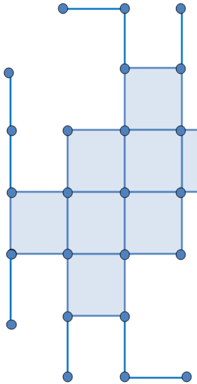
A small piece: (continuous model)



Motivation: moving robots.

Let's build a map of all possible positions. A complete example:

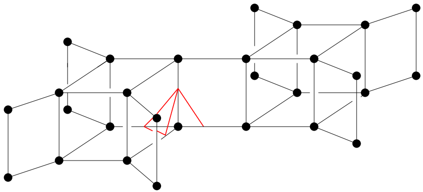


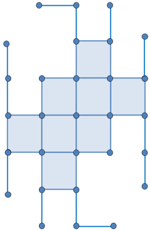


A CAT(0) cube complex!

How can we understand them? Navigate them?

Motivation: moving robots.





How can we understand CAT(0) cube complexes?
How should we navigate them?

Obstacles:

- High dimension.
- Complicated ramification.
- Too many vertices.

This is what we need to overcome.

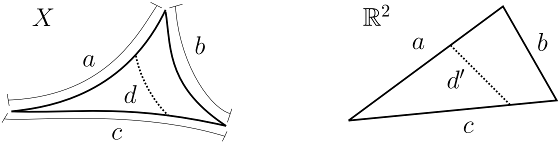
2. PRELIMINARIES. CAT(0) spaces

A metric space X is CAT(0) if it has non-positive curvature everywhere, in the sense that triangles in X are “thinner" than flat triangles. Roughly, it is “saddle shaped".

More precisely, we require:

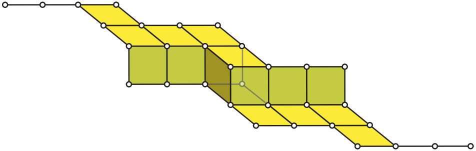
- There is a unique geodesic path between any two points of X.
- (CAT(0) inequality) Consider any triangle T in X and a *comparison triangle* T' of the same sidelengths in the Euclidean plane $\mathbb{R}^2$. Consider any chord (of length d) in T and the corresponding chord (of length d') in T'. Then

$$d \leq d'.$$



PRELIMINARIES. Cube complexes

A cube complex is a space obtained by gluing cubes (of possibly different dimensions) along their faces.

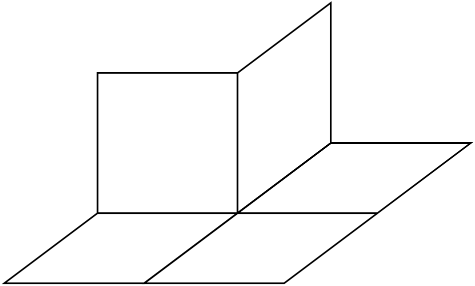


(Like a simplicial complex, but the building blocks are cubes.)

Metric: Euclidean inside each cube.

We are interested in cube complexes which are CAT(0).

Example. Five squares glued around a corner.



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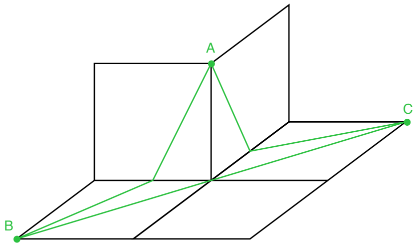

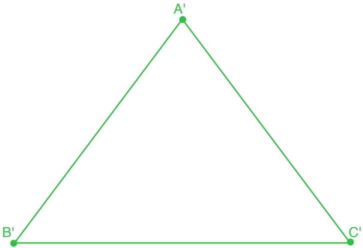

$$|AB| = |AC| = \sqrt{5} \qquad \rightarrow \qquad |A'B'| = |A'C'| = \sqrt{5}$$

$$|BC| = 2\sqrt{2} \qquad\qquad |B'C'| = 2\sqrt{2}$$

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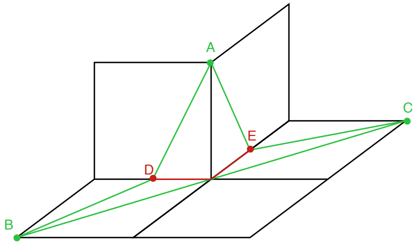
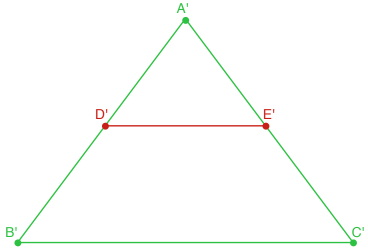




$$|AB| = |AC| = \sqrt{5} \quad \to \quad |A'B'| = |A'C'| = \sqrt{5}$$
$$|BC| = 2\sqrt{2} \qquad\qquad |B'C'| = 2\sqrt{2}$$

$$|DE| = 1 \quad < \quad \sqrt{2} = |D'E'|.$$

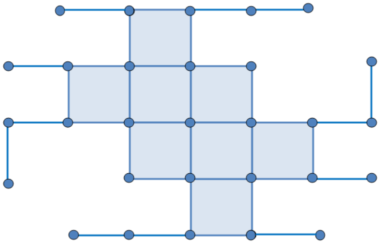
This triangle is thin. (But: I still need to test many chords.)
This space is CAT(0). (But: I still need to test many triangles.)

Not so practical!

3. EXAMPLES.

Example 1. Robot motion planning

State complex. vertices = positions. edges = moves.
cubes = “physically independent" moves.

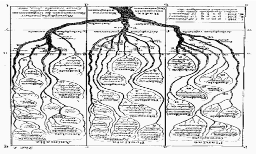


Theorem (GP) This is often a CAT(0) cube complex.

This works **very** generally for many reconfiguration systems, where we change vertex labels on a graph using local moves.

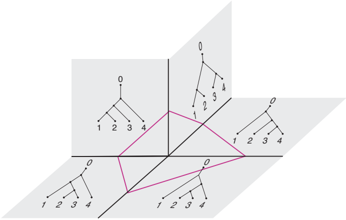
Example 2. Phylogenetic trees (Billera, Holmes, Vogtmann):

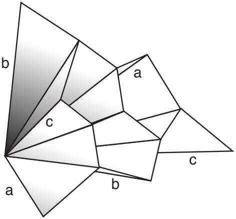
Goal: Predict the evolutionary tree of
n current-day species/languages/....



Approach:

- Build a space $\mathbf{T}_n$ of all possible trees.
- Study it, navigate it.





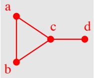
Thm. (BHV) $\mathbf{T}_n$ is a CAT(0) cube complex.	**Cor.** $\mathbf{T}_n$ has unique geodesics. **Cor.** “Average" trees exist.

Example 3. Geometric Group Theory.

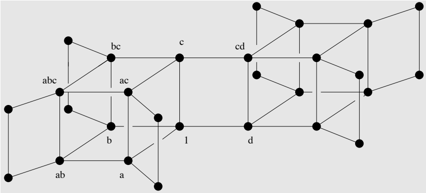
A right-angled Coxeter group is a group of the form

$$W(G) = \langle v \in V \mid v^2 = 1 \text{ for } v \in V, (uv)^2 = 1 \text{ for } uv \in E \rangle$$

Example: $a^2 = b^2 = c^2 = d^2 = 1$
$(ab)^2 = (ac)^2 = (bc)^2 = (cd)^2 = 1$



Thm. (Davis) Right-angled Coxeter groups are CAT(0):
$W(G)$ acts "very nicely" on a CAT(0) cube complex $X(G)$.



Use the geometry of $X(G)$ to study the group $W(G)$; *e.g.*,

- If a group G is CAT(0), the "word problem" is easy for G.

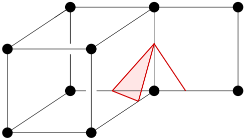
4. CHARACTERIZATIONS.

Which cube complexes are CAT(0)?

In general, CAT(0) is a subtle condition; but for cube complexes:

1. Gromov's characterization.

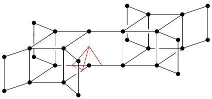
Theorem. (Gromov, 1987)
A cube complex is CAT(0) if and only if it is simply connected and the link of every vertex is a flag simplicial complex.



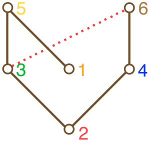
Δ flag: if the 1-skeleton of a simplex T is in Δ, then T is in Δ.

Characterizations: Which cube complexes are CAT(0)?

2. Our characterization.



Theorem. (A-Owen-Sullivant 08)
(Pointed) CAT(0) cube complexes are in bijection with posets with inconsistent pairs.



PIP: A poset P and a set of “inconsistent pairs" $\{x, y\}$, with

$$x, y \text{ inconsistent}, \quad y < z \quad \rightarrow \quad x, z \text{ inconsistent}.$$

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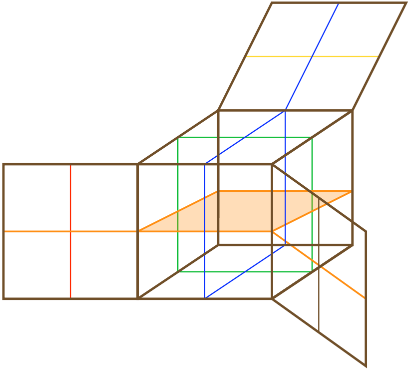
Sketch of proof.

Idea: CAT(0) cube complexes “look like" distributive lattices.
So imitate Birkhoff’s bijection: distributive lattices $\leftrightarrow$ posets

“ $\rightarrow$ ”: X has hyperplanes which split cubes in half. (Sageev)



Theorem. (A. - Owen - Sullivant 08)
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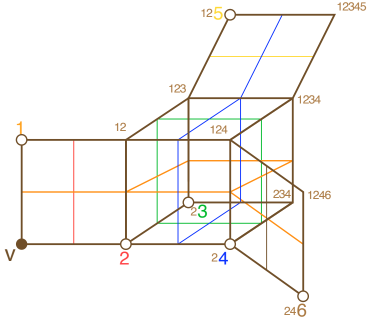
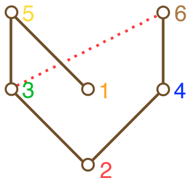
Bijection. " $\rightarrow$ ": Fix a "home" vertex v.





If i, j are hyperplanes, declare:

$i < j$ if one needs to cross i before crossing j
i, j inconsistent if it is impossible to cross them both.

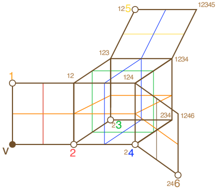
Remark.

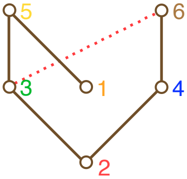
Sageev (95) and Roller (98) obtained a different combinatorial description. Which one is more useful depends on the context.

Let's see some applications.

Application 1. Embeddability conjecture.

Conjecture. (Niblo, Sageev, Wise) Any d-dimensional interval in a CAT(0) cube complex can be embedded in the cubing $\mathbb{Z}^d$.





Proof. (AOS 08)

Dilworth already showed (in 1950!) how to embed $J(Q)$ in $\mathbb{Z}^d$:

- Write Q as a union of d disjoint chains. (Example: 246, 35, 1)
- "Straighten" the cube complex along each chain. $\square$

(Proof also by Brodzki, Campbell, Guentner, Niblo, Wright (08).)

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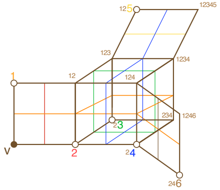

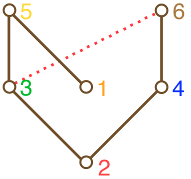

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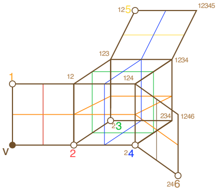
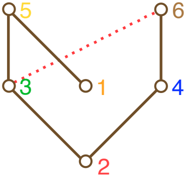
Application 2. All CAT(0) cube complexes are "robotic".

Theorem. (Ghrist-Peterson 07)
Every CAT(0) cube complex can be realized as a state complex.

Their proof is indirect.





Alternative proof. (AOS 10)
Root X $\rightarrow$ poset with inconsistent pairs P.

A "virus robot" takes over the poset P.
It can take over a new cell q if and only if:

o it already took over all elements $p < q$, **and**
o it hasn't taken over any elements inconsistent with q.

Then X is the state complex for this robot. □

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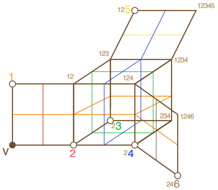

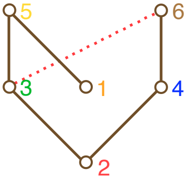

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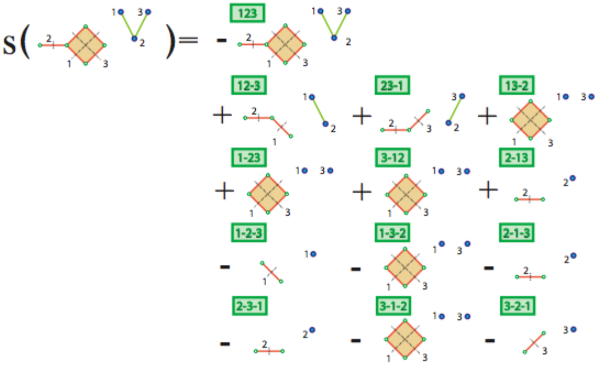
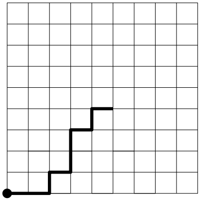
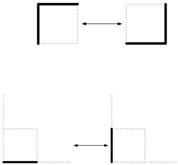
Application 3. The Hopf algebra of CAT(0) cube complexes.

Theorem. (A. - Cifuentes - Collazos 12)
CAT(0) cube complexes have the structure of a Hopf algebra.
There is an elegant formula for the antipode.

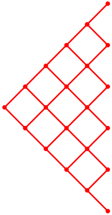


Application 4.1. Pinned-down robotic arm in a square grid.





Theorem. (A.-Baker-Yatchak, 2012) The state complex is a CAT(0) cubical complex. Its PIP (“remote control") is as shown.



Complex of 2^n states in $\frac{n}{2}$ dim. $\longrightarrow \sim \frac{1}{2}n^2$ “buttons".

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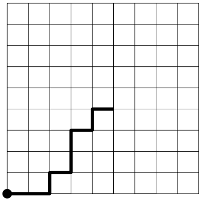
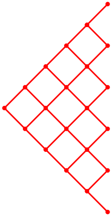
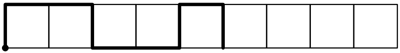


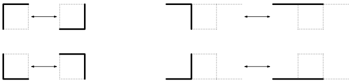


Corollary. (A.-Baker-Yatchak, 2012) Let $q_{n,d}$ be the number of d-cubes in the state complex for the robotic arm of length n. Then

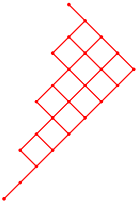
$$\sum_{n,d\geq 0} q_{n,d}\, x^n y^d = \frac{1+xy}{1-2x-x^2y}.$$

Application 4.2. Pinned-down robotic arm in a strip.



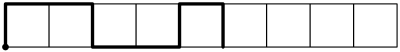


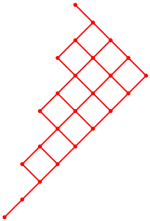
Theorem. (A.-Baker-Yatchak, 2012) The state complex is a CAT(0) cubical complex. Its PIP ("remote control") is as shown.



Complex of $F_n \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ states in $\frac{n}{3}$-dim $\longrightarrow \sim \frac{n^2}{4}$ buttons.

Application 4.2. Pinned-down robotic arm in a strip

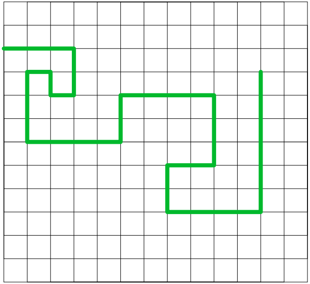




Corollary. (A.-Baker-Yatchak, 2012) Let $s_{n,d}$ be the number of d-cubes in the state complex for the robotic arm of length n. Then

$$\sum_{n,d\geq 0} s_{n,d}\, x^n y^d = \frac{1+x+xy+x^2y}{1-x-x^2-x^3y}.$$

Application 4.3. Non-pinned-down robotic snake.



A negative result:

Theorem. (A. - Yatchak, 2012) If the snake in a grid is not pinned down, the state complex is not always CAT(0).

Open question. Which robots give CAT(0) cube complexes?

Application 5. Moving (some) robots efficiently.

Motivation:

Algorithm. (Owen-Provan 09) A polynomial-time algorithm to find the geodesic between trees T_1 and T_2 in the space of trees $\mathbf{T}_n$.

($\sqrt{2}$-approx.: Amenta 07, exp.: GeoMeTree 08, GeodeMaps 09)

This allows us to

- find distances between trees
- “average" trees.

Application 5. Moving (some) robots efficiently.

We use the PIP ("remote control") of X to get:

Algorithm. (A. - Owen - Sullivant 12, A - Baker - Yatchak 14)
An algorithm to find the geodesic between points p and q in **any** CAT(0) cube complex X.

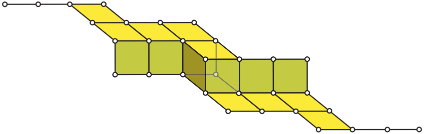
We do this for four metrics:

- Euclidean length
- Time
- Number of moves.
- Number of sets of simultaneous moves.

This allows us to

- navigate the state complex of any reconfiguration system
- find the optimal robot motion between two positions.

(Computer/robotic implementation?)



much as gr acias

Los primeros dos artículos y esta presentación están en:

Advances in Applied Mathematics **48** (2012) 142-163.
SIAM J. Discrete Math. **28-2** (2014), pp. 986-1007
http://arxiv.org/abs/1101.2428
http://arxiv.org/abs/1211.1442
http://math.sfsu.edu/federico