Deformations and Derived Equivalence over Symmetric Algebras

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joint-work with

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A Quick Introduction to Quivers and Path Algebras

Let $\Bbbk$ be an algebraically closed field.

- A **quiver** Q is a directed graph $Q = (Q_0, Q_1, s, e)$ where Q_0 is the set of vertices, Q_1 is the set of arrows and $s, e : Q_1 \to Q_0$ are maps such that for any arrow $\alpha \in Q_1$, $s(\alpha)$ is the vertex where α starts and $e(\alpha)$ is the vertex where α ends.

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- Let $i, j \in Q_0$. A **path** of length $l \geq 1$ from i to j is a composition of arrows $\alpha_l \alpha_{l-1} \cdots \alpha_1$ such that $s(\alpha_1) = i$, $e(\alpha_k) = s(\alpha_{k+1})$ for all k with $1 \leq k \leq l-1$ and $e(\alpha_l) = j$.

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- We also define for any vertex i of Q a path of length zero (from i to itself), denoted by e_i.

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- We also define for any vertex i of Q a path of length zero (from i to itself), denoted by e_i.
- The **path algebra** $\Bbbk Q$ of Q is defined to be the k-vector space with $\Bbbk$-basis the set of all paths in Q and the product of two paths is taken to be the composition if it exists, and zero otherwise.

Example

Consider the quiver

$$Q = \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet} \xrightarrow{\gamma} \underset{4}{\bullet}$$

- $Q_0 = \{\dot{1}, \dot{2}, \dot{3}, \dot{4}\}$ and $Q_1 = \{\alpha, \beta, \gamma\}$. Note that $s(\alpha) = \dot{1}, e(\alpha) = \dot{2} = s(\beta)$.
- Then $\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \beta\alpha, \gamma\beta, \gamma\beta\alpha\}$ is a $\mathbb{k}$-basis of the path algebra $\mathbb{k}Q$.
- Note that $\alpha\gamma = 0 = \alpha\beta$. Since $\beta\alpha \neq \alpha\beta$ then in particular $\mathbb{k}Q$ is not a commutative $\mathbb{k}$-algebra.

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Theorem 1 (Morita). *If Λ is a finite dimensional $\mathbb{k}$-algebra, then there is a unique basic algebra Λ_0 up to isomorphism with $\Lambda \sim_M \Lambda_0$. We call Λ_0 the basic algebra of Λ.*

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Theorem 2 (Gabriel). *Any basic finite dimensional $\mathbb{k}$-algebra is of the form $\mathbb{k}Q/I$ for a unique quiver Q and some ideal I with $J^n \subseteq I \subseteq J^2$ for some $n \geq 2$, where J is the ideal of $\mathbb{k}Q$ generated by all arrows of Q.*

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(iii) *Λ is of* **wild type***, i.e., there are infinitely many isomorphism classes of indecomposable Λ-modules, and Λ-mod is comparable with $\Bbbk\langle x, y\rangle$-mod.*

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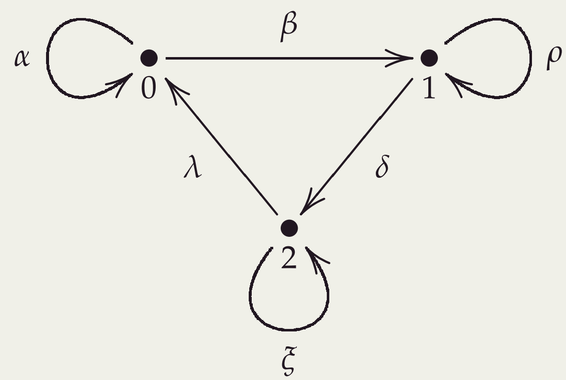
When Λ is special biserial $\Bbbk$-algebra then all the indecomposable non-projective Λ-modules can be described combinatorially from Q and I using so-called strings and bands. We call the former **string Λ-modules** and the latter **band Λ-modules**(M.C.R BUTLER & C.M. RINGEL, 1987).

A particular example

Let $\Lambda_3 = \Bbbk Q/I$, where

$Q =$ 

(1)

and

$$I = \langle \beta\alpha, \rho\beta, \delta\rho, \xi\delta, \lambda\xi, \alpha\lambda, \lambda\delta\beta - \alpha^2, \beta\lambda\delta - \rho^2, \delta\beta\lambda - \xi^2 \rangle \tag{2}$$

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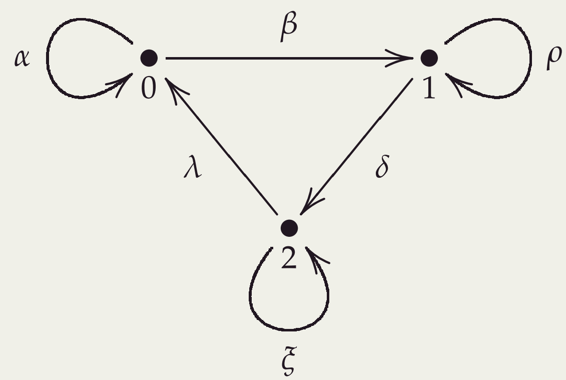
$Q =$ 

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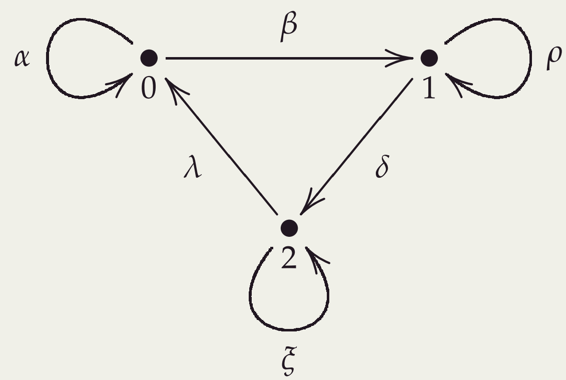
$Q =$ 

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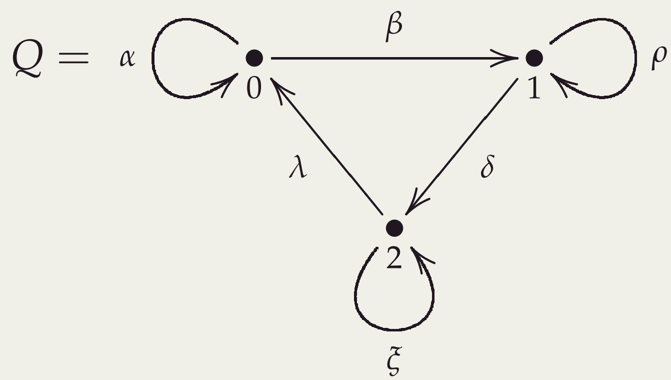

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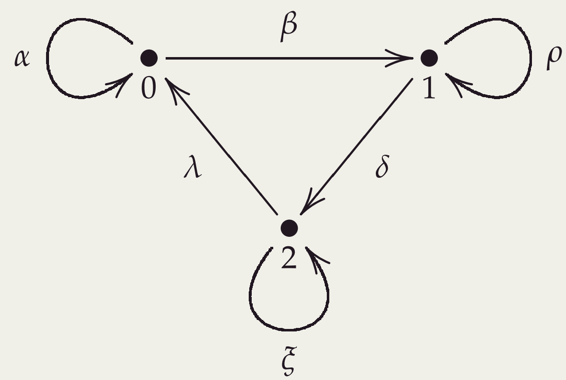

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- two 3-tubes (consisting entirely of string modules);
- infinitely many 1-tubes (consisting entirely of band modules).

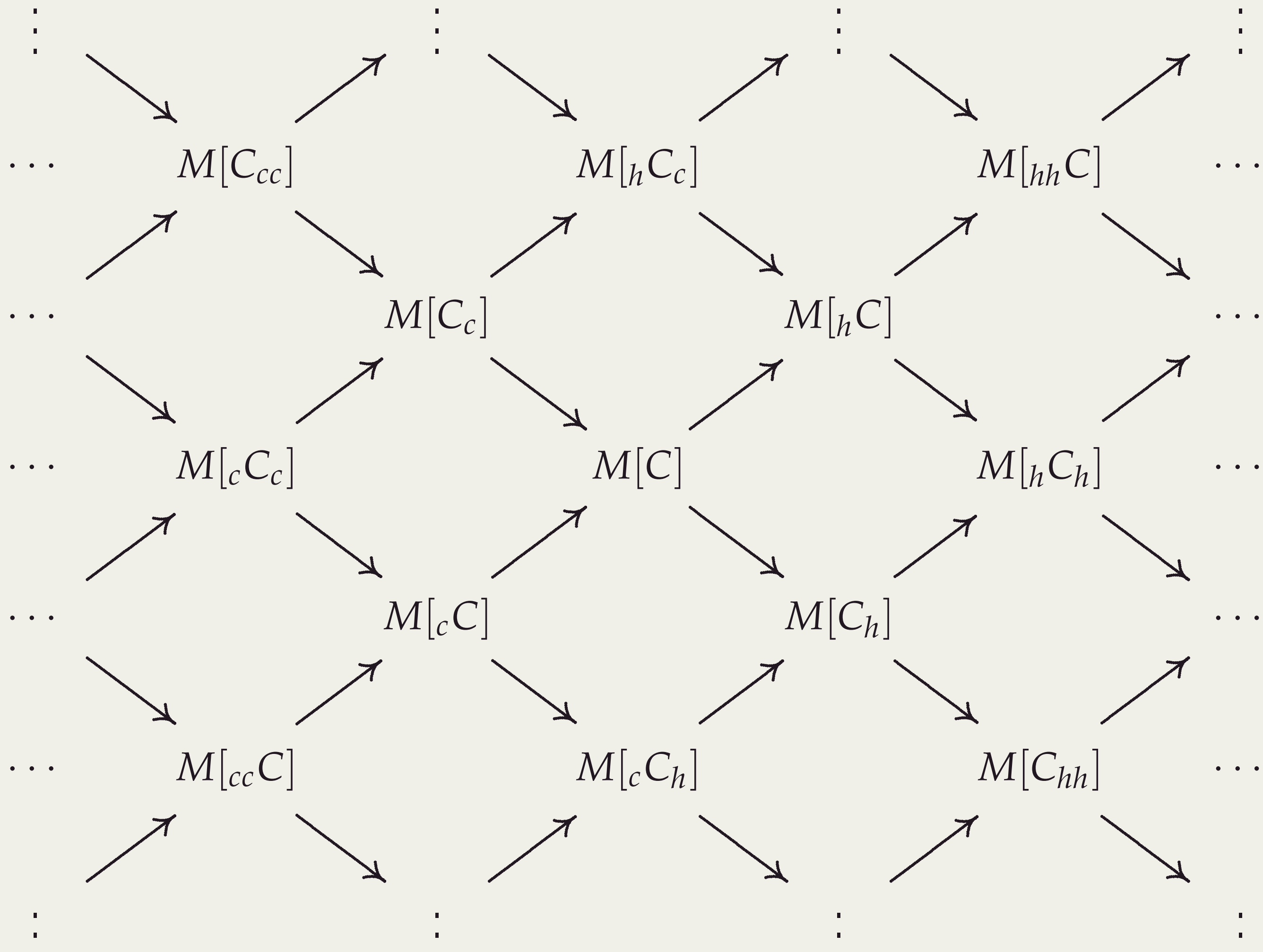
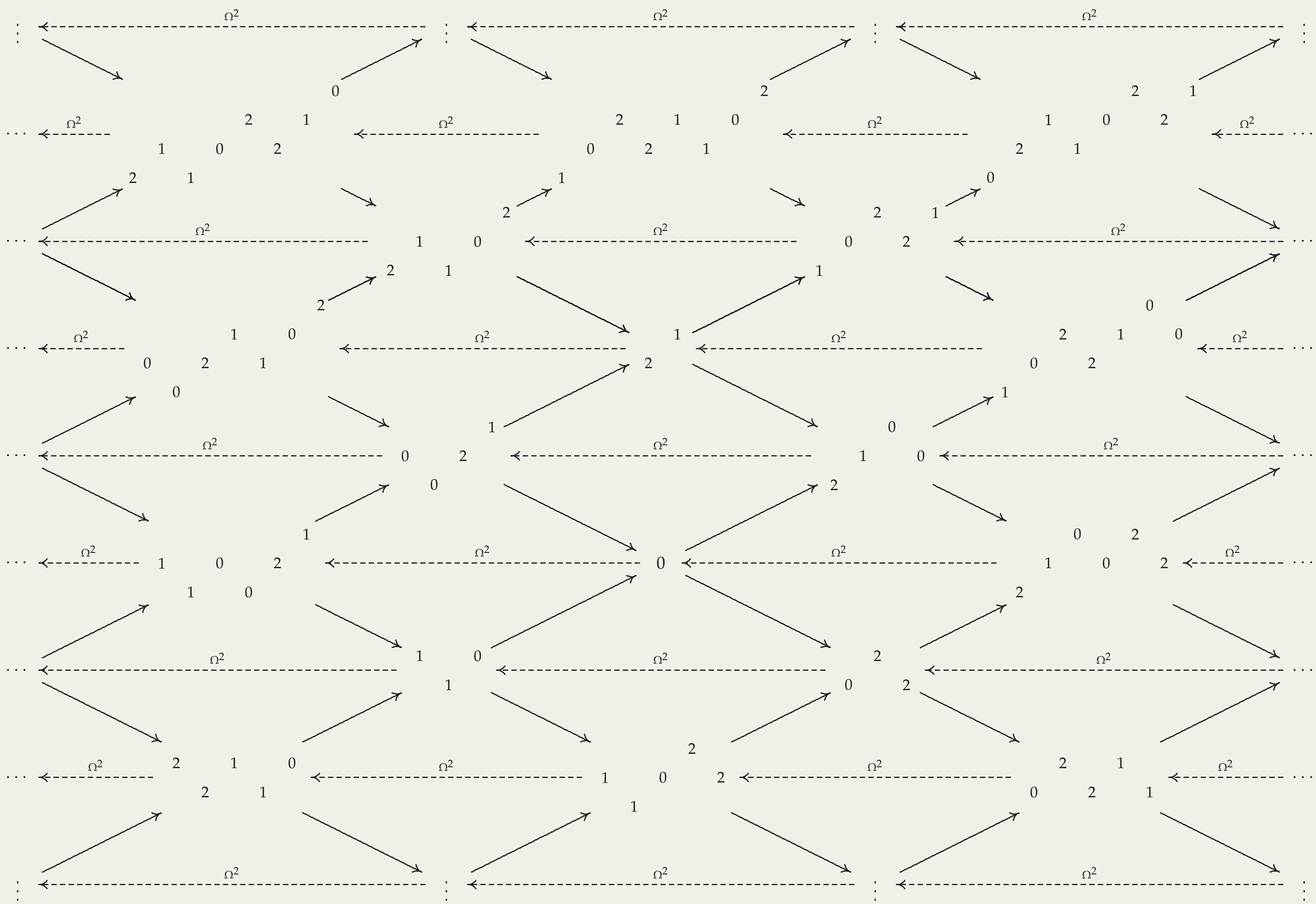


Figure 1: The stable Auslander-Reiten component of type $\mathbb{Z}\mathbb{A}_\infty^\infty$ near $M[C]$.



$\underline{\mathrm{End}}_\Lambda(M)$	$\mathrm{Ext}^1_\Lambda(M,M)$	$R(\Lambda,M)$
$\ncong k$	–	–
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$\cong k$	k	$k[[t]]$
$\cong k$	0	k
$\cong k$	k	$k[[t]]/(t^2)$
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Figure 2: The stable Auslander-Reiten component of type $\mathbb{Z}\mathbb{A}_\infty^\infty$ near S_0.

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- Λ is an arbitrary finite-dimensional $\Bbbk$-algebra equipped with the discrete topology.
- For all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, we denote by $R\Lambda$ the tensor product of $\Bbbk$-algebras $R \otimes_{\Bbbk} \Lambda$. Note in particular that $R\Lambda$ is also a $\Bbbk$-vector space.

Deformations and Derived Categories

Let $\Bbbk$ be an algebraically closed field, and let G be a profinite group.

- In the 1980's, B. MAZUR developed a deformation theory of finite dimensional representations of G over $\Bbbk$. His work was based on that of M. SCHLESSINGER-1968. A more explicit approach was latter described by B. DE SMITH and H.W. LENSTRA in the year 1995.

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- Deformation theory has become a basic tool in arithmetic geometry (see e.g. CORNELL, G., SILVERMAN, J.H., and STEVENS, G. (Eds.), "**Modular Forms and Fermat's Last Theorem**", Springer-Verlag, 1997, and its references).
- The main motivation of this talk is that powerful tools from representation theory of finite dimensional algebras, such as Auslander-Reiten quivers, stable equivalences, and combinatorial descriptions of modules has been used to have a better understanding of the deformation theory of group representations.
- This approach has lead to the solution of various open problems. For example, in 2006 F. BLEHER and T. CHINBURG successfully used this approach to construct representations whose universal deformation ring is not a complete intersection.

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Definition 4. An $R\Lambda$-module M is said to be **pseudocompact** provided that it is the projective limit of $R\Lambda$-modules of finite length having the discrete topology. We denote by $\mathrm{PCMod}(R\Lambda)$ the abelian category of pseudocompact $R\Lambda$-modules.

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Definition 5. Let M (resp. N) be a left (resp. right) pseudocompact R-module. The **complete tensor product** of M and N is a pseudocompact R-module $M\hat{\otimes}_R N$ and a R-bilinear map $\theta : M \times N \to M\hat{\otimes}_R N$ with the following property: given *any* R-bilinear map $f : M \times N \to L$, where L is a pseudocompact R-module, there exists a unique morphism of pseudocompact R-modules $g : M\hat{\otimes}_R N \to L$ such that $g\theta = f$.

See work of P. GABRIEL and A. BRUMMER to get more details about pseudocompact modules.

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- We say that a complex $M^\bullet$ in $K^-(\mathrm{PCMod}(R\Lambda))$ has **finite pseudocompact R-tor dimension**, if there exists an integer N such that for all pseudocompact R-modules S, and for all integers $i < N$, $\mathrm{H}^i(S\hat{\otimes}_R^{\mathbf{L}}M^\bullet) = 0$, where $\hat{\otimes}_R^{\mathbf{L}}$ denotes the left derived functor of $\hat{\otimes}_R$.

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Lemma 7. *A complex $M^\bullet \in \mathrm{Ob}(K^-(\mathrm{PCMod}(R\Lambda))$ has finite pseudocompact R-tor dimension if and only if there exists a complex $P^\bullet \in \mathrm{Ob}(K^b(\mathrm{PCMod}(R\Lambda))$ whose terms are topologically free R-modules such that $P^\bullet$ is quasi-isomorphic to $M^\bullet$.*

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Example 8. If M is a pseudocompact $R\Lambda$-module, we regard M as a complex concentrated in dimension 0. It follows that M has finite pseudocompact R-tor dimension if and only if M is a free R-module.

Quasi-lifts, Deformations and the Deformation Functor

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- Two quasi-lifts of $(M^\bullet, \phi)$ and $(M'^\bullet, \phi')$ of $V^\bullet$ over R are **isomorphic** if there exists an isomorphism $f : M^\bullet \to M'^\bullet$ in $D^-(\mathrm{PCMod}(R\Lambda))$ such that $\phi' \circ (\Bbbk\hat{\otimes}^{\mathbf{L}}_R f) = \phi$.

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  - For all morphism $\alpha : R \to R'$ in $\hat{\mathcal{C}}$, $\hat{\mathrm{F}}_{V^\bullet}(\alpha)$ is the set map $\hat{\mathrm{F}}_{V^\bullet}(R) \to \hat{\mathrm{F}}_{V^\bullet}(R')$ such that for all deformation $[M^\bullet, \phi]$ of $V^\bullet$ over R, $\hat{\mathrm{F}}_{V^\bullet}(\alpha)([M^\bullet, \phi]) = [M'^\bullet, \phi']$, where $M'^\bullet = R'\hat{\otimes}^{\mathbf{L}}_{R,\alpha} M^\bullet$ and ϕ' is the composition

$$\Bbbk\hat{\otimes}^{\mathbf{L}}_{R'} M' = \Bbbk\hat{\otimes}^{\mathbf{L}}_{R'} (R'\hat{\otimes}^{\mathbf{L}}_{R,\alpha} M^\bullet) \cong \Bbbk\hat{\otimes}^{\mathbf{L}}_R M^\bullet \xrightarrow{\phi} V^\bullet.$$

Versal (Universal) Deformation Rings

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(i) The functor $\mathrm{F}_{V^\bullet}$ *has a pro-representable hull* $R(\Lambda, V^\bullet)$ *in the sense of Schlessinger, and the functor* $\hat{\mathrm{F}}_{V^\bullet}$ *is continuous, i.e., for all* $R \in \mathrm{Ob}(\hat{\mathcal{C}})$*,*

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$$h : t_{\mathrm{F}_{V^\bullet}} \to \mathrm{Ext}^1_{D^-(\mathrm{PCMod}(\Lambda))}(V^\bullet, V^\bullet) = \mathrm{Hom}_{D^-(\mathrm{PCMod}(\Lambda))}(T(V^\bullet), V^\bullet),$$

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(iii) *If* $\mathrm{Hom}_{D^-(\mathrm{PCMod}(\Lambda))}(V^\bullet, V^\bullet) = \Bbbk$, *then* $\hat{\mathrm{F}}_{V^\bullet}$ *is represented by* $R(\Lambda, V^\bullet)$.

Remark 12. *1. By Theorem 11 (i), there exists a deformation* $[U(\Lambda, V^\bullet), \phi_{U(\Lambda,V^\bullet)}]$ *of* $V^\bullet$ *over* $R(\Lambda, V^\bullet)$ *with the following property. For each* $R \in \mathrm{Ob}(\hat{\mathcal{C}})$*, the map* $\mathrm{Hom}_{\hat{\mathcal{C}}}(R(\Lambda, V^\bullet), R) \to \hat{\mathrm{F}}_{V^\bullet}(R)$ *induced by* $\alpha \mapsto R\hat{\otimes}_{R(\Lambda,V^\bullet),\alpha} U(\Lambda, V^\bullet)$ *is surjective, and this map is bijective if* R *is the ring of dual numbers* $\Bbbk[\epsilon]$ *over* $\Bbbk$*, where* $\epsilon^2 = 0$*. The ring* $R(\Lambda, V^\bullet)$ *and the deformation* $[U(\Lambda, V^\bullet), \phi_{U(\Lambda,V^\bullet)}]$ *are uniquely determined up to non-canonical isomorphism. In this situation, we call* $R(\Lambda, V^\bullet)$ *the* **versal deformation ring** *of* $V^\bullet$*.*

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2. If $R(\Lambda, V^\bullet)$ *represents* $\hat{\mathrm{F}}_{V^\bullet}$, *we call* $R(\Lambda, V^\bullet)$ *the* **universal deformation ring** *of* $V^\bullet$.

Corollary 13. *If* $V^\bullet$ *consists of a single* Λ*-module* V_0 *of finite dimension over* $\Bbbk$, *then the versal deformation ring* $R(\Lambda, V^\bullet)$ *coincides with the versal deformation ring* $R(\Lambda, V_0)$ *studied by* F. BLEHER & J. V-M. *in 2012.*

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More precisely, consider diagrams of Artinian objects in $\hat{\mathcal{C}}$ of the form

$$\begin{array}{ccccc} & & D & & \\ & \swarrow^{\alpha'} & & \searrow^{\beta'} & \\ A & & & & B \\ & \searrow_{\alpha} & & \swarrow_{\beta} & \\ & & C & & \end{array} \tag{3}$$

where $D = A \times_C B$ and β is a surjective small extension, i.e., the kernel of β is a principal ideal $tB \cong \Bbbk$ that is annihilated by $\mathfrak{m}_B$. For each such diagrams, consider the natural map of pullbacks

$$\chi_{V^\bullet} : \mathrm{F}_{V^\bullet}(D) \longrightarrow \mathrm{F}_{V^\bullet}(A) \times_{\mathrm{F}_{V^\bullet}(C)} \mathrm{F}_{V^\bullet}(B). \tag{4}$$

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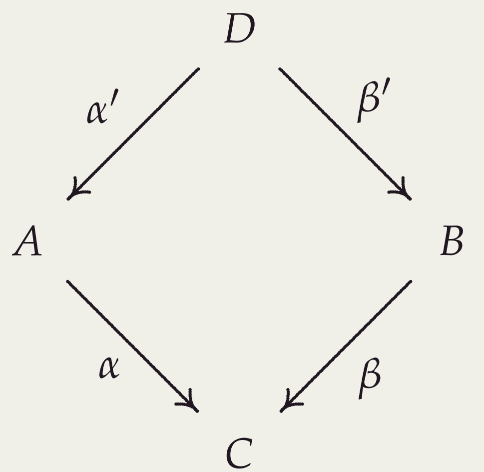

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- We have a short exact sequence

$$0 \to \epsilon M^\bullet \xrightarrow{\iota_{M^\bullet}} M^\bullet \xrightarrow{\pi_{M^\bullet}} M^\bullet/\epsilon M^\bullet \to 0 \tag{5}$$

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- The mapping cone of $\iota_{M^\bullet}$ is $C(\iota_{M^\bullet})^\bullet = T(\epsilon M^\bullet) \oplus M^\bullet$ with i-th differential

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- We obtain a triangle in $K^-(\mathrm{PCMod}(\Bbbk[\epsilon]\Lambda))$

$$\epsilon M^\bullet \xrightarrow{\iota_{M^\bullet}} M^\bullet \xrightarrow{g} C(\iota_{M^\bullet})^\bullet \xrightarrow{f_{M^\bullet}} T(\epsilon M^\bullet), \tag{6}$$

- We then get a triangle in $K^-(\mathrm{PCMod}(\Bbbk[\epsilon]\Lambda))$

$$\begin{array}{ccccccc}
C(\iota_{M^\bullet})^\bullet & \xrightarrow{f_{M^\bullet}} & T(\epsilon M^\bullet) & \longrightarrow & C(f_{M^\bullet})^\bullet & \longrightarrow & T(C(\iota_{M^\bullet})^\bullet) \\
\downarrow{\scriptstyle (0,\pi_{M^\bullet})} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle T(0,\pi_{M^\bullet})} \\
M^\bullet/\epsilon M^\bullet & & T(\epsilon M^\bullet) & & T(M^\bullet) & & T(M^\bullet/\epsilon M^\bullet)
\end{array} \tag{7}$$

where the downward arrows are quasi-isomorphisms in $C^-(\mathrm{PCMod}(\Bbbk[\epsilon]\Lambda))$.

The tangent space $t_{\mathrm{F}_{V^\bullet}}$ (Cont.)

- We then get a triangle in $K^-(\mathrm{PCMod}(\Bbbk[\epsilon]\Lambda))$

$$
\begin{array}{ccccccc}
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- Hence the diagram

$$
\begin{array}{ccccc}
 & & C(\iota_{M^\bullet})^\bullet & & \\
 & {\scriptstyle (0,\pi_{M^\bullet})}\swarrow & & \searrow{\scriptstyle f_{M^\bullet}} & \\
M^\bullet/\epsilon M^\bullet & & & & T(\epsilon M^\bullet)
\end{array} \tag{8}
$$

defines a morphism $\hat{f}_{M^\bullet} : M^\bullet/\epsilon M^\bullet \to T(\epsilon M^\bullet)$ in $D^-(\mathrm{PCMod}(\Bbbk[\epsilon]\Lambda))$.

- Thus, we get a triangle in $D^-(\mathrm{PCMod}(\Bbbk[\epsilon]\Lambda))$:

$$M^\bullet/\epsilon M^\bullet \xrightarrow{\hat{f}_{M^\bullet}} T(\epsilon M^\bullet) \longrightarrow T(M^\bullet) \longrightarrow T(M^\bullet/\epsilon M^\bullet) \tag{9}$$

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$$M^\bullet/\epsilon M^\bullet \xrightarrow{\hat{f}_{M^\bullet}} T(\epsilon M^\bullet) \longrightarrow T(M^\bullet) \longrightarrow T(M^\bullet/\epsilon M^\bullet) \tag{9}$$

- Using the isomorphism $\phi : M^\bullet/\epsilon M^\bullet \to V^\bullet$ in $D^-(\mathrm{PCMod}(\Bbbk[\epsilon]\Lambda))$, we obtain a morphism

$$\hat{f}_{M^\bullet,1} \in \mathrm{Hom}_{D^-(\mathrm{PCMod}(\Bbbk[\epsilon]\Lambda))}(V^\bullet, T(V^\bullet))$$

associated to $\hat{f}_{M^\bullet}$, where $\hat{f}_{M^\bullet,1}$ is as in the diagram (10).

$$\begin{array}{ccccccc}
M^\bullet/\epsilon M^\bullet & \xrightarrow{\hat{f}_{M^\bullet}} & T(\epsilon M^\bullet) & \longrightarrow & T(M^\bullet) & \to & T(M^\bullet/\epsilon M^\bullet) \\
\phi \big\downarrow & & \big\downarrow T(\psi) & & & & \\
V^\bullet & & T(M^\bullet/\epsilon M^\bullet) & & & & \\
& \overset{\hat{f}_{M^\bullet,1}}{\searrow} & \big\downarrow T(\phi) & & & & \\
& & T(V^\bullet) & & & &
\end{array} \tag{10}$$

- We get an association $\hat{h}$ defined by

$$\hat{h}: \quad \mathrm{F}_{V^\bullet}(\Bbbk[\epsilon]) \to \mathrm{Hom}_{D^-(\mathrm{PCMod}(\Bbbk[\epsilon]\Lambda))}(V^\bullet, T(V^\bullet)) \tag{11}$$

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- We prove that $\hat{h}$ is an isomorphism of $\Bbbk$-vector spaces.

Versal (Universal) Deformation Rings and Derived Equivalence

Recall that for any ring S, we denote by S-mod the category of finitely generated left S-modules.

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Theorem 14 (J. RICKARD, 1991). *The $\Bbbk$-algebras Λ and Γ are derived equivalent if and only if there is a bounded complex $P^\bullet$ of finitely generated $\Lambda - \Gamma$-bimodules and a bounded complex $Q^\bullet$ of finitely generated $\Lambda - \Gamma$-bimodules such that*

$$P^\bullet \otimes^{\mathbf{L}}_\Gamma Q^\bullet \cong \Lambda \qquad \text{in } D^b((\Lambda \otimes_\Bbbk \Lambda^{\text{op}})\text{-mod}), \text{ and} \tag{12}$$

$$Q^\bullet \otimes^{\mathbf{L}}_\Lambda P^\bullet \cong \Gamma \qquad \text{in } D^b((\Gamma \otimes_\Bbbk \Gamma^{\text{op}})\text{-mod}).$$

If $P^\bullet$ and $Q^\bullet$ exists, then the functors

$$P^\bullet \otimes^{\mathbf{L}}_\Gamma - : D^b(\Gamma\text{-mod}) \to D^b(\Lambda\text{-mod}) \qquad \text{and} \tag{13}$$

$$Q^\bullet \otimes^{\mathbf{L}}_\Lambda - : D^b(\Lambda\text{-mod}) \to D^b(\Gamma\text{-mod})$$

are equivalences of derived categories, and $Q^\bullet$ is isomorphic to $\mathbf{R}\mathrm{Hom}_\Lambda(P^\bullet, \Lambda)$ in the derived category of $\Gamma - \Lambda$-bimodules.

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If Λ and Γ are derived equivalent $\Bbbk$-algebras, then we say that the complexes $P^\bullet$ and $Q^\bullet$ in Theorem 14 are called **two-sided tilting complexes**.

Definition 15. A finite-dimensional $\mathbb{k}$-algebra Λ is said to be **symmetric**, provided that Λ and $\Lambda^* = \mathrm{Hom}_{\mathbb{k}}(\Lambda, \mathbb{k})$ are isomorphic as $\Lambda - \Lambda$-bimodules.

Corollary 16 (J. RICKARD, 1996)**.** *Let Λ and Γ be* **symmetric** *finite dimensional $\mathbb{k}$-algebras. Then Λ and Γ are derived equivalent if and only if there exists a bounded complex $P^\bullet$ of finitely generated $\Lambda - \Gamma$-bimodules such that all of the terms of $P^\bullet$ are projective as left and right modules and such that*

$$\Lambda \cong P^\bullet \otimes_\Gamma (P^\bullet)^* \cong \mathrm{Hom}_\Gamma(P^\bullet, P^\bullet) \quad \text{in } K^b((\Lambda \otimes_{\mathbb{k}} \Lambda^{op})\text{-mod}), \text{ and} \tag{14}$$

$$\Gamma \cong (P^\bullet)^* \otimes_\Lambda P^\bullet \cong \mathrm{Hom}_\Lambda(P^\bullet, P^\bullet) \quad \text{in } K^b((\Gamma \otimes_{\mathbb{k}} \Gamma^{op})\text{-mod}),$$

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$$\begin{aligned} \Lambda &\cong P^\bullet \otimes_\Gamma (P^\bullet)^* \cong \mathrm{Hom}_\Gamma(P^\bullet, P^\bullet) && \text{in } K^b((\Lambda \otimes_{\Bbbk} \Lambda^{op})\text{-mod}), \text{ and} \\ \Gamma &\cong (P^\bullet)^* \otimes_\Lambda P^\bullet \cong \mathrm{Hom}_\Lambda(P^\bullet, P^\bullet) && \text{in } K^b((\Gamma \otimes_{\Bbbk} \Gamma^{op})\text{-mod}), \end{aligned} \tag{14}$$

where $(P^\bullet)^* = \mathrm{Hom}_{\Bbbk}(P^\bullet, \Bbbk)$.

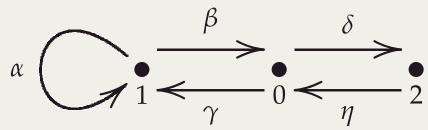
Definition 17. RICKARD calls a complex $P^\bullet$ as in Corollary 16 a **split-endomorphism two-sided tilting complex**.

The following result is proved similarly to the work of F. BLEHER on "**Deformations and derived equivalence**" in 2006.

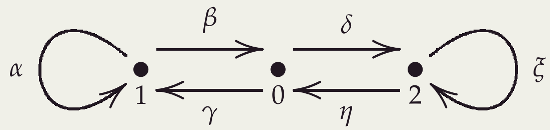
Theorem 18. *Let Λ and Γ be symmetric finite dimensional $\Bbbk$-algebras, and let $Q^\bullet$ be a split-endomorphism two-sided tilting complex in $D^b(\Gamma \otimes_{\Bbbk} \Lambda^{op}\text{-mod})$. Let $V^\bullet$ be a bounded complex of finitely generated Λ-modules, and let $V'^\bullet = Q^\bullet \otimes_\Lambda V^\bullet$. Then $R(\Lambda, V^\bullet)$ and $R(\Gamma, V'^\bullet)$ are isomorphic.*

An Example: Four Algebras of Dihedral Type

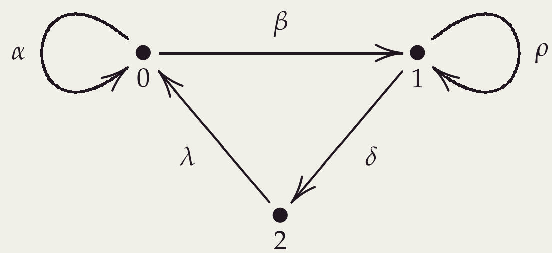
Consider the algebras Λ_0, Λ_1, Λ_2, and Λ_3, where

$\Lambda_0 = D(3\mathscr{B})_2^{2,2,2} = \Bbbk[$


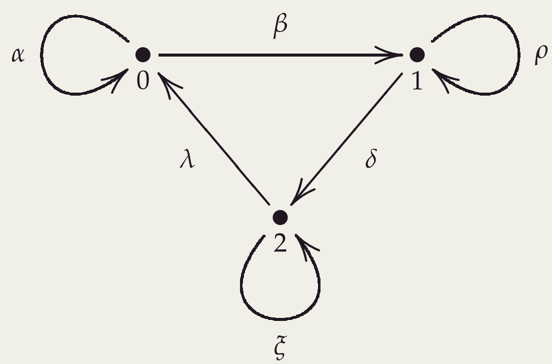
$]/\langle \alpha\gamma, \beta\alpha, \delta\beta, \gamma\eta, (\beta\gamma)^2 - (\eta\delta)^2, (\gamma\beta)^2 - \alpha^2 \rangle$

$\Lambda_1 = D(3\mathscr{D})_2^{1,2,2,2} = \Bbbk[$


$]/\langle \alpha\gamma, \beta\alpha, \delta\beta, \gamma\eta, \xi\delta, \eta\xi, \gamma\beta - \alpha^2, (\delta\eta)^2 - \xi^2, \beta\gamma - (\eta\delta)^2 \rangle$

$\Lambda_2 = D(3\mathscr{Q})^{2,2,2} = \Bbbk[$


$]/\langle \beta\alpha, \alpha\lambda, \rho\beta, \delta\rho, (\lambda\delta\beta)^2 - \alpha^2, (\beta\lambda\delta)^2 - \rho^2 \rangle$

$\Lambda_3 = D(3\mathscr{R})^{1,2,2,2} = \Bbbk[$


$]/\langle \beta\alpha, \rho\beta, \delta\rho, \xi\delta, \lambda\xi, \alpha\lambda, \lambda\delta\beta - \alpha^2, \beta\lambda\delta - \rho^2, \delta\beta\lambda - \xi^2 \rangle$

THE ALGEBRAS $D(3\mathscr{B})_2^{2,2,2}$, $D(3\mathscr{D})_2^{1,2,2,2}$, $D(3\mathscr{Q})^{2,2,2}$ AND $D(3\mathscr{R})^{1,2,2,2}$ (CONT.)

- The algebras Λ_0, Λ_1, Λ_2 and Λ_3 are all $\Bbbk$-algebras of **dihedral type** (K. ERDMANN, 1990), hence they are **symmetric** $\Bbbk$-algebras.

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- The isomorphism classes of the universal deformation rings of finitely generated Λ_3-modules V with $\underline{\mathrm{End}}_{\Lambda_3}(V) = \Bbbk$ lying in a connected component of the stable Auslander-Reiten quiver of Λ_3 have been completely classified by F. M. BLEHER & J.V-M in 2012. The universal deformation rings are either isomorphic to $\Bbbk$, or to $\Bbbk[[t]]/(t^2)$, or to $\Bbbk[[t]]$.

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Lemma 19 (T. HOLM, 1999). *The $\Bbbk$-algebras $\Lambda_0 = D(3\mathscr{B})_2^{2,2,2}$, $\Lambda_1 = D(3\mathscr{D})_2^{1,2,2,2}$, $\Lambda_2 = D(3\mathscr{Q})^{2,2,2}$ and $\Lambda_3 = D(3\mathscr{R})^{1,2,2,2}$ are derived equivalent.*

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Remark 20. *Although the algebras Λ_0, Λ_1, Λ_2 and Λ_3 are derived equivalent, they are not Morita equivalent.*

The Algebras $D(3\mathscr{B})_2^{2,2,2}$, $D(3\mathscr{D})_2^{1,2,2,2}$, $D(3\mathscr{Q})^{2,2,2}$ and $D(3\mathscr{R})^{1,2,2,2}$ (Cont.)

By Corollary 16, for all $i \in \{0,1,2\}$, there is a split-endomorphism two-sided tilting complex $Q_i^\bullet$ in $D^b(\Lambda_i \otimes_{\Bbbk} \Lambda_3^{\mathrm{op}} - \mathrm{mod})$ that realizes the derived equivalence in Lemma 19.

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Hence we obtain (by Theorem 18) that for every bounded complex $V^\bullet$ of finitely generated Λ_3-modules,

$$R(\Lambda_3, V^\bullet) \cong R(\Lambda_i, Q^\bullet \otimes_{\Lambda_3} V^\bullet).$$

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Since derived equivalences induce stable equivalences, we get the following result:

Theorem 21. *Let* $\Lambda \in \{D(3\mathscr{B})_2^{2,2,2}, D(3\mathscr{D})_2^{1,2,2,2}, D(3\mathscr{Q})^{2,2,2}, D(3\mathscr{R})^{1,2,2,2}\}$, *and let* V *a* Λ*-module such that* $\underline{\mathrm{End}}_\Lambda(V) = \Bbbk$ *lying in a connected component of the stable Auslander-Reiten quiver of* Λ*. Then, the universal deformation ring* $R(\Lambda, V)$ *of* V *is isomorphic either to* $\Bbbk$*, or to* $\Bbbk[[t]]/(t^2)$*, or to* $\Bbbk[[t]]$.

GRACIAS