Twisted partial actions of groups on semilattices of groups

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All the semilattices of groups we consider contain identity.

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If A is a semilattice of groups, then we assume that $E(A) = \langle 1_x \mid x \in G \rangle$.

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If such a partial homomorphism $\Gamma : G \to S$ exists, then (i) guarantees that S is inverse.

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  - (iv) $f'(\phi(s), \phi(t)) = g(s)\lambda_s(g(t))f(s,t)g(st)^{-1}$, $s, t \in S$.

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Remark

If A is inverse, then $A *_\Theta G$ is inverse.

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making $A *_\Theta G$ to be an extension of A by S.

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Two partial extensions $(\Gamma : G \to S, A \xrightarrow{i} U \xrightarrow{j} S)$ and $(\Gamma' : G \to S', A \xrightarrow{i'} U' \xrightarrow{j'} S')$ of A by G are called equivalent

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such that the following diagrams commute:

$$\begin{tikzcd} G \arrow[r, "\Gamma"] \arrow[dr, "\Gamma'"'] & S \arrow[d, "\nu"] \\ & S' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, equal] & U \arrow[r, "j"] \arrow[d, "\mu"] & S \arrow[d, "\nu"] \\ A \arrow[r, "i'"] & U' \arrow[r, "j'"] & S' \end{tikzcd}$$

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Theorem

Any partial extension of A by G is equivalent to $A *_{\Theta} G$ for some twisted partial action Θ of G on A.

References

DOKUCHAEV, M., EXEL, R., AND SIMÓN, J. J.
Crossed products by twisted partial actions and graded algebras.
J. Algebra 320, 8 (2008), 3278–3310.

DOKUCHAEV, M., EXEL, R., AND SIMÓN, J. J.
Globalization of twisted partial actions.
Trans. Amer. Math. Soc. 362, 8 (2010), 4137–4160.

EXEL, R.
Partial actions of groups and actions of inverse semigroups.
Proc. Amer. Math. Soc. 126, 12 (1998), 3481–3494.

LAUSCH, H.
Cohomology of inverse semigroups.
J. Algebra 35 (1975), 273–303.

THANK YOU!