# The $\phi$-dimension: A new homological measure

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Let $A$ be an artin $R$-algebra. We denote by $mod\,(A)$ the category of finitely generated left $A$-modules We denote by $add(M)$ the category of all direct summands of (finite) direct sums of $M$.

$\underline{mod}\,(A)$ is the stable $R$-category modulo projectives, whose objects are the same as in $mod\,(A)$ and the morphisms are given by $\underline{Hom}_A(M, N) := Hom_A(M, N)/\mathcal{P}(M, N)$, where $\mathcal{P}(M, N)$ is the $R$-submodule of $Hom_A(M, N)$ consisting of the morphisms $M \to N$ factoring through objects in $proj\,(A)$.

# The finitistic dimension conjecture(Bass -1960)

$$fin.dim(A) = sup\{pd(M) | M \in mod(A) \text{ with } pd(M) < \infty\} < \infty$$

The class of Igusa-Todorov algebras contain many others, for example: algebras of representation dimension at most 3, algebras with radical cube zero, monomial algebras and left serial algebras. In fact, it is expected that all artin algebras are Igusa-Todorov.

The interest in the finitistic dimension is because of the "finitistic dimension conjecture", which is still open, and states that *the finitistic dimension of any artin algebra is finite.* This conjecture is closely related with several homological conjectures, and therefore it is a centrepiece for the development of the representation theory of artin algebras.

# Igusa-Todorov function ($\phi : \mathrm{Obj}\,(mod\,(A)) \to \mathbb{N}$)

Let $K(A)$ denote the quotient of the free abelian group generated by the set of iso-classes $\{[M] \;:\; M \in mod(A)\}$ modulo the relations:

(a) $[N] - [S] - [T]$ if $N \simeq S \oplus T$ and

(b) $[P]$ if $P$ is projective.

The syzygy functor $\Omega : \underline{mod}\,(A) \to \underline{mod}\,(A)$ gives rise to a group homomorphism $\Omega : K(A) \to K(A)$, where $\Omega([M]) := [\Omega(M)]$.

Let $\langle M \rangle = \langle add(M) \rangle$ denote the $\mathbb{Z}$-submodule of $K(A)$ generated by the indecomposable non-projective direct summands of $M$.

Since the rank of $\Omega(\langle M \rangle)$ is less or equal than the rank of $\langle M \rangle$, which is finite, it follows from the well ordering principle that there exists the smallest non-negative integer $\phi(M)$ such that $\Omega : \Omega^n(\langle M \rangle) \to \Omega^{n+1}(\langle M \rangle)$ is an isomorphism for all $n \geq \phi(M)$.

## Lemma

*Let $A$ be an artin $R$-algebra and $M, N \in mod(A)$. Then, the following statements hold.*

(a) $\phi(M) = pd\, M$ *if* $pd\, M < \infty$.

(b) $\phi(M) = 0$ *if* $M$ *is indecomposable and* $pd\, M = \infty$.

(c) $\phi(M) \leq \phi(N \oplus M)$.

(d) $\phi(M) = \phi(N)$ *if* $add\,(M) = add\,(N)$.

(e) $\phi(M \oplus P) = \phi(M)$ *for any* $P \in proj\,(A)$.

(f) $\phi(M) \leq \phi(\Omega\, M) + 1$.

It follows, from the above properties, that $\phi$ is a good refinement of the measure "projective dimension". Indeed, for modules of finite projective dimension both homological measures coincides; and in the case of infinite projective dimension, $\phi$ gives a finite number as a measure.

## Exemple

# Global dimension

**Definition**

$gldim(A) = sup\{pdM : M \in mod(A)\}$

$gldim(A) = 0$ if and only if semisimple algebra;
$gldim(A) = 1$ if and only if hereditary algebra.

# The $\phi$-dimension

**Definition**

$$\phi dim(A) = sup\{\phi(M) : M \in mod(A)\}$$

$$fin.dim(A) \leq \phi dim(A) \leq gldim(A)$$

Recall that a algebra A is self-injective if seen as the left R-module is injective. Note that every indecomposable module over a self-injective algebra is either projective or has infinite projective dimension. However, this property does not characterise self-injective rings.

**Theorem (Huard, Lanzilotta)**

*Let A be an artin R-algebra. The algebra A is self-injective if and only if* $\phi dim(A) = 0$

**Exemple**

$$fin.dim(A) < \phi dim(A) < gldim(A)$$

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# $\phi$-dimension and the bi-functors $Ext_A^i(-,-)$

$$pdM = m \text{ se e somente se } Ext^{m+1}(M,-) = 0 \; e \; Ext^m(M,-) \neq 0$$

Denotamos por $C_A$ a categoria abeliana de todos os $R$-funtores $F : mod(A) \to mod(R)$. $[F] := \{G \in C_A : G \simeq F\}$.

**Proposition (-,Lanzilotta, Mendonça)**

*Let $A$ be an artin $R$-algebra and $M, N \in mod(A)$. Then, the following conditions are equivalent.*

(a) $[\Omega^n M] = [\Omega^n N]$ *in* $K(A)$.
(b) $Ext_A^i(M,-) \simeq Ext_A^i(N,-)$ *in* $C_A$ *for any* $i \geq n+1$.
(c) $Ext_A^{n+1}(M,-) \simeq Ext_A^{n+1}(N,-)$ *in* $C_A$.

The proof follows the Auslander-Reiten formulas and Yoneda's Lemma

### Definition

*Let $A$ be an artin $R$-algebra, $d$ be a positive integer and $M$ in $\operatorname{mod}(A)$. A pair $(X, Y)$ of objects in $\operatorname{add}(M)$ is called a d-Division of $M$ if the following three conditions hold:*

(a) $\operatorname{add}(X) \cap \operatorname{add}(Y) = \{0\}$,

(b) $Ext^d_A(X, -) \not\simeq Ext^d_A(Y, -)$ *in* $C_A$,

(c) $Ext^{d+1}_A(X, -) \simeq Ext^{d+1}_A(Y, -)$ *in* $C_A$.

### Theorem (-,Lanzilotta, Mendonça)

*Let $A$ be an artin $R$-algebra and $M$ in $\operatorname{mod}(A)$. Then*

$$\phi(M) = max(\{d \in \mathbb{N} : \text{ there is a d-Division of } M\} \cup \{0\}).$$

## Remark

*Observe that $\phi(M) = 0$ if and only if for any pair $(X, Y)$ of objects in add $(M)$, which are not projective and add $(X) \cap$ add $(Y) = \{0\}$, we have that $Ext^d_A(X, -) \not\approx Ext^d_A(Y, -)$ in $C_A$ for any $d \geq 1$. Thus, in this case, the following set is empty*

$$\{d \in \mathbb{N} \ : \ \textit{there is a d-Division of } M\}.$$

For any non-negative integer $i$ and any $M, N \in \mod(A)$, it is well known the existence of an isomorphism

$$\operatorname{Tor}_i^A(D(M), N)) \simeq D\, Ext_A^i(N, M),$$

which is natural in both variables. Hence, a $d$-Division can be given in terms of Tor's functors as follows.

**Remark**

*Let $A$ be an artin $R$-algebra, $d$ be a positive integer and $M$ in $\mod(A)$. Then, a pair $(X, Y)$ of objects in $add\,(M)$ is a $d$-Division of $M$ if the following three conditions hold:*

(a) $add\,(X) \cap add\,(Y) = \{0\}$;

(b) $\operatorname{Tor}_d^A(-, X) \not\simeq \operatorname{Tor}_d^A(-, Y)$ *in* $C_{A^{op}}$;

(c) $\operatorname{Tor}_{d+1}^A(-, X) \simeq \operatorname{Tor}_{d+1}^A(-, Y)$ *in* $C_{A^{op}}$.

As a consequence of Remark and Theorem, we get that the Igusa-Todorov function $\Phi$ can also be characterised by using the Tor's bi-functors $\operatorname{Tor}_i^A(-,-)$.

**Theorem (-, Lanzilotta, Mendonça)**

*Let $A$ and $B$ be artin algebras, which are derived equivalent. Then, $\phi dim\,(A) < \infty$ if and only if $\phi dim\,(B) < \infty$. More precisely, if $T^\bullet$ is a tilting complex over $A$ with $n$ non-zero terms and such that $B \simeq End_{D\,(A)}(T^\bullet)$, then*

$$\phi dim\,(A) - n \leq \phi dim\,(B) \leq \phi dim\,(A) + n.$$

# Generalisation of the classic Bongartz's-1981

Following Y. Miyashita in, it is said that an $A$-module $T \in$ mod $(A)$ is a tilting module, if $T$ satisfies the following properties:

(1) $pdT$ is finite

(2) $Ext^i_A(T, T[i]) = 0$ for $i \neq 0$.

(3) there is an exact sequence $0 \to {}_AA \to T_0 \to T_1 \to \cdots \to T_m \to 0$ in $mod\,(A)$, with $T_i \in add\,(T)$ for any $0 \leq i \leq m$.

**Corollary**

*Let $A$ be an artin algebra, and let $T \in$ mod $(A)$ be a tilting $A$-module. Then, for the artin algebra $B := End_A(T)^{op}$, we have that*

$$\phi dim\,(A) - pd\ T \leq \phi dim\,(B) \leq \phi dim\,(A) + pd\ T.$$

## Corollary

*Let $A$ and $B$ be two finite-dimensional $k$-algebras, $M \in mod\,(A)$ and $N \in mod\,(B)$. Let $A[M]$ and $B[N]$ be the respective one-point extensions. If $A$ and $B$ are derived-equivalent, then the finiteness of the $\phi$-dimension of one of the algebras $A, B, A[M]$ and $B[N]$ implies that all of them have finite $\phi$-dimension.*

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