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Pseudoclassical mechanics *à la* Faddeev–Jackiw

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In this study, we propose an extension of the formulation developed by Faddeev and Jackiw to include anticommutative variables in the language of supergeometry, as it could be a cost-effective way to determine the ($\mathbb{Z}_2$-graded) Poisson structure of theories describing spin-like degrees of freedom. Specifically, we apply the developed approach to pseudoclassical systems to lately use the standard canonical quantization program to check whether their already known quantum description is recovered.

Keywords: Faddeev-Jackiw approach; pseudoclassical mechanics; Grassmann variables; canonical quantization.

1. Introduction

The advent of quantum theory undoubtedly taught physicists that the well-known laws that govern the everyday observable world are just a limiting case of a more general and elegant theory, which has proven to be one of the greatest and most profound achievements of human knowledge — understand science — about the understanding of nature in its most primitive phase. It is also well known that there are quantum systems that have classical analogue and this feature paved the way for the powerful tool known as quantization, which states that one can establish

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a one-to-one correspondence from the mathematical framework of the classical description of a system to the mathematical structure of its corresponding quantum depiction. In this sense, the so-called quantization is merely an ambitious attempt to start from a particular theory and arrive at a more general one. Nevertheless, the enormous success of the correspondence principle suggests that in practice it is possible, at least formally, to make use of this sort of inverse recipe of building an acceptable quantum modeling from a previously established classical theory.

Another new concept brought by quantum mechanics that revolutionized our view of nature at atomic scales is that each constituent of matter carries an intrinsic[a] angular momentum known as spin. Our current knowledge of spin is that it is always quantized and its corresponding quantum number can take either integer or half-integer values; particles with integer spin are called bosons, while particles with half-integer spin are known as fermions, and they behave very differently in nature. According to Feynman, even though the rules for spin have been easily stated, no one has found a simple explanation for them, which suggests that our understanding of spin is still incomplete.[1] Mathematically, the quantum fields used to describe bosonic particles obey commutation relations, whilst anticommutation relations govern fermionic fields. Provided that a field theory can be thought of as an infinite degrees of freedom extension of the description of a mechanical system, we might expect that a *classical* description of spin degrees of freedom inherit that anticommutative behavior; therefore, the coordinates describing the spin degrees of freedom should be Grassmann variables. The theory that takes into account the mentioned ingredient is known as pseudoclassical mechanics or pseudomechanics after the name was coined by Casalbuoni,[2,3] though Berezin and Marinov also made enormous contributions to the subject.[4,5] It was shown that pseudomechanics is the classical limit ($\hbar \to 0$) of a general nonrelativistic quantum theory with both Bose and Fermi operators. In this sense, a pseudoclassical system admits the possibility of describing the spin at a pre-quantum level.

It is also important to mention that any classical system described by a Lagrangian on configuration space can be classified as regular or singular through the Hessian criterion, which implies that any Lagrangian linear in time-derivatives of generalized coordinates is always singular and so the standard Hamiltonian construction is not well-defined; therefore, a consistent canonical analysis of the theory requires the use of Dirac method[6] to obtain a dynamics compatible with the constraints involved. Nevertheless, there is another equivalent formulation developed by Faddeev and Jackiw[7] that allows to treat a Lagrangian linear in velocities as nonsingular. This approach is built on the kinematically independent variables of the standard phase space (in canonical formulation); notwithstanding, we will refer to the set of these variables also as phase space. In fact, in the Faddeev–Jackiw (FJ) formulation, the Lagrangian is written in such a way that it is linear in *velocities*.[b]

[a]In the sense that it has nothing to do with its motion in space.

[b]In this case, we use this term to refer to the time derivatives of the Faddeev–Jackiw phase space coordinates.

On the other hand, it turns out that any Lagrangian that considers anticommutative variables is necessarily linear in the time derivatives of them and thus, the aforementioned Hessian criterion tells us that such a Lagrangian is singular. However, since this criterion is restricted just to configuration variables and their derivatives, it does not apply to a Lagrangian defined on phase space. Therefore, the Hessian criterion does not forbid the nonsingularity of the particular structure of the Lagrangian employed in the FJ approach. The FJ formulation has some advantages over the Dirac one, among which highlights the lack of need to classify the constraints. However, the main feature of the FJ approach is that it provides a systematic way to obtain the correct Poisson structure for both constrained and unconstrained systems, which is the starting point of the canonical quantization program.

This paper is devoted to the investigation of pseudoclassical discrete systems in the Faddeev–Jackiw formulation since it allows treating this type of systems as nonsingular. The structure of this paper is the following: In Sec. 2, we present the principal ideas and objects involved in the Faddeev–Jackiw pseudoclassical construction for regular systems and their relation with the corresponding classical $\mathbb{Z}_2$-graded brackets. Section 3 contains some instructive examples of pseudomechanical systems and some relevant consequences, such as total angular momentum as Noether charge, and the very primitive idea of supersymmetry (at the pseudoclassical level). In Sec. 4, we proceed to perform the canonical quantization program for the systems presented in the previous section. Finally, Sec. 5 contains our conclusions of this work.

2. Faddeev–Jackiw Theory Involving Grassmann Variables

In this section, we present an extension of the Faddeev–Jackiw approach for discrete systems described by both commutative and anticommutative variables; the latter refers to the so-called Grassmann numbers. Given this, the whole construction is performed on a set with $\mathbb{R}^{m|n}$-structure, which we call *phase superspace*, M, and we denote the corresponding supercoordinates by $\{z^A\}_{A=1}^{m+n}$. It is noteworthy that such an extension has already been developed, but in a subtly different way;[c] see for example Refs. 8 and 9. The Faddeev–Jackiw theory is constructed for a Lagrangian written in its canonical form; i.e. linear in the velocities as we show below[d]:

$$L(z;\dot{z}) = \dot{z}^A\theta_A(z) - H(z), \tag{1}$$

in which H is the Hamiltonian, depending only on z-variables (involving no time-derivatives of them). The coefficients of the velocities are the components of a 1-form

[c] The Faddeev–Jackiw formulation is commonly treated in the component language; however, it is more convenient (from a computational point of view) to deal with the geometric objects involved rather than their components. Furthermore, being careful with the conventions employed in the $\mathbb{Z}_2$-case is of utmost importance.

[d] Throughout this paper, we will use the left-derivative convention.

on M:

$$\theta = dz^A \theta_A, \quad \theta_A = \theta_A(z) \tag{2}$$

called *canonical* 1*-form*. Note that the Lagrangian takes objects of $\mathbb{R}^{2m|2n}$ and assuming it as a homogeneous superfunction, in principle,[e] there are two possibilities for its Grassmann-parity. Then, one can label it by a parameter such that $|L| = \epsilon$. As a consequence, it is possible to deduce from (1):

$$|H| = \epsilon, \quad |\theta_A| = |A| + \epsilon; \tag{3}$$

in which we have denoted $|A| \equiv |z^A|$. Recall that there are two conventions for the exterior derivative operator on superspace, which depends on its Grassmann-parity, namely, Deligne convention ($|d| = 0$) and the Bernstein–Leites ($|d| = 1$) one. For now, we write $|d| = k$, allowing also the two possibilities.[10,f] The next step will be to determine the dynamics generated by the Lagrangian in (1) using Hamilton's principle, that is, by varying the action,

$$S[z] = \int_{t_1}^{t_2} L(z; \dot{z}) dt \tag{4}$$

and obtaining a necessary condition for its extreme value, considering suitable boundary conditions. Then, one arrives at the following equations of motion:

$$\omega_{AB} \dot{z}^B = \frac{\partial_L H}{\partial z^A}, \tag{5}$$

with

$$\omega_{AB} = (-1)^{|B|(\epsilon+1)} \left(\frac{\partial_L \theta_B}{\partial z^A} - (-1)^{|A||B|} \frac{\partial_L \theta_A}{\partial z^B} \right). \tag{6}$$

From (6) it is straightforward to check the following properties:

$$\begin{cases} \omega_{AB} = (-1)^{(|A|+1)(|B|+1)+\epsilon(|A|+|B|)} \omega_{BA}, \\ |\omega_{AB}| = |A| + |B| + \epsilon. \end{cases} \tag{7}$$

It is interesting to see what happens if we take the exterior derivative[g] of the canonical 1-form:

$$d\theta = \frac{1}{2} (-1)^{1+k|B|} dz^B \wedge dz^A \left(\frac{\partial_L \theta_B}{\partial z^A} - (-1)^{|A||B|} \frac{\partial_L \theta_A}{\partial z^B} \right). \tag{8}$$

[e] Although all models of physical interest are described by an even Lagrangian, for our purposes it is useful to consider both cases.

[f] Given a p-form α and a q-form β, in the Deligne convention, $\alpha \wedge \beta = (-1)^{pq+|\alpha||\beta|} \beta \wedge \alpha$ and $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, whereas in the Bernstein–Leites (BL) convention, $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$ and $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$.[11] On the other hand, the BL convention can be interpreted also as an even derivation but with different gradings: $\alpha \wedge \beta = (-1)^{(|\alpha|+p)(|\beta|+q)} \beta \wedge \alpha$ and $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|+p} \alpha \wedge d\beta$, see for example.[12]

[g] In our convention, the exterior derivative acts on the left.

Then, from (6) it is possible to write

$$d\theta = -\frac{1}{2}(-1)^{|B|(k+\epsilon+1)} dz^B \wedge dz^A \omega_{AB}, \tag{9}$$

thus, picking conveniently $k+\epsilon = 1$, we obtain

$$d\theta = -\frac{1}{2} dz^B \wedge dz^A \omega_{AB}. \tag{10}$$

Therefore, with this choice for $k = k(\epsilon)$, we get the following fundamental relation:

$$d\theta = -\omega. \tag{11}$$

On the other hand, the Grassmann-parity of the 2-form[h] ω is presented as follows:

$$\begin{aligned} |\omega| &= \left|\frac{1}{2} dz^B \wedge dz^A \omega_{AB}\right| \\ &= (|B|+\epsilon) + (|A|+\epsilon) + (|A|+|B|+\epsilon) = \epsilon. \end{aligned} \tag{12}$$

From this, we see that the parity of ω coincides with the Lagrangian's one. In the nonsingular case, ω is a symplectic form and its inverse, with components ω^{AB} defined by

$$\omega^{AC}\omega_{CB} = \delta^A_B = \omega_{BC}\omega^{CA} \tag{13}$$

are such that

$$\begin{cases} \omega^{AB} = (-1)^{1+(|A|+\epsilon)(|B|+\epsilon)}\omega^{BA}, \\ |\omega^{AB}| = |A| + |B| + \epsilon. \end{cases} \tag{14}$$

By using (13) we can rewrite the equations of motion (5) as

$$\dot{z}^A = \omega^{AB}\frac{\partial_L H}{\partial z^B}. \tag{15}$$

As usual, one introduces the graded Poisson brackets (GPB) by the requirement:

$$\dot{F} \doteq \{F; H\}_\epsilon, \tag{16}$$

where F is an arbitrary homogeneous dynamical superfunction on M. From this demand, and using (15), we obtain the explicit expression of the GPB of two dynamic superfunctions:

$$\{F; G\}_\epsilon \doteq \frac{\partial_R F}{\partial z^A}\omega^{AB}\frac{\partial_L G}{\partial z^B}, \tag{17}$$

in which the ϵ-dependence of the GPB is due to the fact that the parity of ω^{AB} depends on the parameter ϵ. Moreover, a quick computation yields

$$|\{\cdot;\cdot\}| = \epsilon = |\omega|. \tag{18}$$

[h] It is important to not confuse the object ω with its components ω_{AB}.

Thus, we see that the Grassmann-parity of the GPB is given by that of the 2-form ω. The graded Poisson brackets obey the following properties:

(1) Grassmann-parity:

$$|\{F;G\}_\epsilon| = |F| + |G| + \epsilon. \tag{19}$$

(2) Graded antisymmetry:

$$\{F;G\}_\epsilon = -(-1)^{(|F|+\epsilon)(|G|+\epsilon)}\{G;F\}_\epsilon. \tag{20}$$

(3) Graded Jacobi's identity:

$$\begin{aligned}(-1)^{(|F|+\epsilon)(|H|+\epsilon)}\{F;\{G;H\}_\epsilon\}_\epsilon + (-1)^{(|G|+\epsilon)(|F|+\epsilon)}\{G;\{H;F\}_\epsilon\}_\epsilon \\ + (-1)^{(|H|+\epsilon)(|G|+\epsilon)}\{H;\{F;G\}_\epsilon\}_\epsilon = 0.\end{aligned} \tag{21}$$

(4) Graded Leibniz's rule:

$$\{F;GH\}_\epsilon = \{F;G\}_\epsilon H + (-1)^{(|F|+\epsilon)|G|}G\{F;H\}_\epsilon. \tag{22}$$

The GPBs with $\epsilon = 0$ are known as super Poisson brackets, whereas GPBs with $\epsilon = 1$ are called odd Poisson brackets or classical antibrackets. Rigorously, the GPBs are graded Lie brackets of degree ϵ such that the map $\{F;\cdot\}_\epsilon$ is a left superderivation with Grassmann-parity $|F| + \epsilon$.

The key ingredient in the Faddeev–Jackiw approach is the connection between the inverse of the symplectic form and the GPB, since from (17) it is possible to derive

$$\{z^A;z^B\}_\epsilon = \omega^{AB}. \tag{23}$$

This result is very relevant because we can obtain the GPP of any pair of dynamical superfunctions from the canonical form of any Lagrangian.

As already mentioned, all the physical models involving both ordinary and Grassmannian degrees of freedom have even Grassmann-parity so we will focus on the $\epsilon = 0$ case, which suggests the use of the Bernstein–Leites convention in this approach.

To conclude this section, it is worth mentioning that there is a particular situation in which a system involving both commutative and anticommutative variables could be singular,[i] *viz.* when the system is endowed with gauge freedom. Although not detailed here, the Faddeev–Jackiw approach also contemplates this case and offers an iterative procedure in which the zero modes of the pre-symplectic form play the role of generators of all constraints; however, at some stage of the algorithm, it is necessary to introduce a subsidiary gauge constraint to finally obtain a nondegenerate symplectic form. For more details, we refer the interested reader to review Ref. 13, for example.

[i]Namely, when the 2-form that follows from (11) is degenerate and so presents zero-modes.

3. Specific Cases

3.1. *Case 1: Pseudoclassical spin degrees of freedom*

The main motivation for incorporating Grassmann variables to describe some kind of degrees of freedom in theoretical physics was undoubtedly the need to classically describe spin in a such way that after quantization we will obtain the already known results. This proposal works wonderfully for the nonrelativistic description of spin. To see that, let us consider a system with a configuration superspace $\mathbb{R}^{3|3}$ whose dynamics is derived from the following Lagrangian:

$$L = \frac{1}{2} m\delta_{ab}\dot{q}^a\dot{q}^b - \frac{i}{2}\delta_{ab}\dot{\theta}^a\theta^b. \tag{24}$$

It is worth mentioning that the presence of odd variables implies that the action is not simply given by the usual expression, $S = \int_{t_1}^{t_2} Ldt$, because the boundary conditions for this kind of variable can be treated in a different fashion. For this particular case, the correct action is instead:

$$S = \int_{t_1}^{t_2} \left[L - \frac{i}{2}\frac{d}{dt}\left(\delta_{ab}\theta^a(t_1)\theta^b(t)\right)\right] dt \tag{25}$$

and the initial conditions demanded to solve the variational problem, according to Hamilton's principle are[14]

$$\delta q^a(t_2) = 0 = \delta q^a(t_1), \quad \delta\theta^a(t_1) + \delta\theta^a(t_2) = 0. \tag{26}$$

To start with the Faddeev–Jackiw scheme, we need to write the Lagrangian (24) in its canonical form; for this purpose, it is convenient to obtain the corresponding canonical momenta:

$$p_a = \frac{\partial L}{\partial \dot{q}^a} = m\delta_{ab}\dot{q}^b, \quad \pi_a = \frac{\partial_L L}{\partial\dot{\theta}^a} = -\frac{i}{2}\delta_{ab}\theta^b. \tag{27}$$

Note that, from (27), there is a dependence between π_a and θ^a, and thus, we choose the following coordinates for the Faddeev–Jackiw phase superspace $\mathbb{R}^{6|3}$:

$$z^{st} = (q^a \quad p_a \quad \theta^a). \tag{28}$$

Then, the canonical Lagrangian is given as follows:

$$L = \dot{q}^a p_a - \frac{i}{2}\delta_{ab}\dot{\theta}^a\theta^b - H \tag{29}$$

with

$$H = \frac{1}{2m}\delta^{ab}p_a p_b. \tag{30}$$

A remarkable feature of this system is that there is no contribution involving the odd variables in the Hamiltonian, which suggests that the time evolution of the θ^a could not be generated by H. From (29), we recognize the canonical 1-form[j]:

$$\vartheta = dq^a p_a - \frac{i}{2}\delta_{ab} d\theta^a \theta_b. \tag{31}$$

Then, the phase superspace is equipped with the 2-form obtained by (11), namely:

$$\omega = \delta^a_b dq^b \wedge dp_a + \frac{i}{2}\delta_{ab} d\theta^b \wedge d\theta^a, \tag{32}$$

whose matrix representation results as

$$\begin{array}{cc} & \begin{array}{ccc} dq^b & dp_b & d\theta^b \end{array} \\ \begin{array}{c} dq^a \\ dp_a \\ d\theta^a \end{array} & \left(\begin{array}{cc|c} 0 & -\delta^b_a & 0 \\ \delta^a_b & 0 & 0 \\ \hline 0 & 0 & i\delta_{ab} \end{array}\right) \end{array}. \tag{33}$$

The supermatrix above turns out to be nonsingular and therefore, its inverse can be written as

$$\begin{array}{cc} & \begin{array}{ccc} \dfrac{\partial}{\partial q^b} & \dfrac{\partial}{\partial p_b} & \dfrac{\partial_L}{\partial \theta^b} \end{array} \\ \begin{array}{c} \dfrac{\partial}{\partial q^a} \\ \dfrac{\partial}{\partial p_a} \\ \dfrac{\partial_L}{\partial \theta^a} \end{array} & \left(\begin{array}{cc|c} 0 & \delta^a_b & 0 \\ -\delta^b_a & 0 & 0 \\ \hline 0 & 0 & -i\delta_{ab} \end{array}\right) \end{array}. \tag{34}$$

Hence, according to (23), we obtain the nonnull graded Poisson brackets of this model:

$$\{q^a; p_b\} = \delta^a_b, \quad \{\theta^a; \theta^b\} = -i\delta^{ab}. \tag{35}$$

It is important to mention that the θ-variables transform under the orthogonal group $O(3)$[3] and so we can perform a $O(3)$-transformation for all coordinates; doing this we find that the Lagrangian (24) remains invariant under such a transformation. Therefore, according to the Noether theorem, there must exist a conserved charge associated with the $O(3)$-symmetry. Considering infinitesimal transformations of the form $R^c{}_a = \delta^c_a + \omega^c{}_a \in O(3)$, with $\omega^c{}_a$ antisymmetric, we get the following small variation for the coordinates:

$$\delta q^c = \omega^c{}_a q^a, \quad \delta\theta^c = \omega^c{}_a \theta^a \tag{36}$$

[j] We are using ϑ to denote the canonical 1-form just to not confuse it with the odd coordinates.

with this, the corresponding Noether charge will be given by[k]

$$\begin{aligned}\omega_{ab}M^{ab} &= -\omega^{c}{}_{a}q^a p_c - \omega^{c}{}_{a}\theta^a\left(-\frac{i}{2}\delta_{cb}\theta^b\right)\\ &= \frac{1}{2}\omega_{ab}(q^a\delta^{bc}p_c - q^b\delta^{ac}p_c - i\theta^a\theta^b).\end{aligned} \tag{37}$$

Defining

$$L^{ab} \doteq q^a\delta^{bc}p_c - q^b\delta^{ac}p_c, \quad S^{ab} \doteq -i\theta^a\theta^b. \tag{38}$$

The Noether charge can be expressed as

$$M^{ab} = L^{ab} + S^{ab}. \tag{39}$$

In order to identify physical quantities it is convenient to construct the following objects:

$$L_a \doteq \frac{1}{2}\epsilon_{abc}L^{bc}, \quad S_a \doteq \frac{1}{2}\epsilon_{abc}S^{bc}; \tag{40}$$

which leads to

$$L_a = \epsilon_{ab}{}^{c}q^b p_c, \quad S_a = -\frac{i}{2}\epsilon_{abc}\theta^b\theta^c. \tag{41}$$

We immediately recognize that the L_a are the components of the orbital angular momentum, and the classical brackets between them are

$$\{L_a; L_b\} = \epsilon_{ab}{}^{c}L_c, \tag{42}$$

whilst it is not hard to show that the S_a is such that

$$\{S_a; S_b\} = \epsilon_{ab}{}^{c}S_c. \tag{43}$$

Note that $|S_a| = 0$ and hence could be a quantity with physical content. On the other hand, the usual canonical quantization procedure shows that the quantum operators $\hat{S}_a$ satisfy the commutation relations for the spin angular momentum. Moreover, note that if we define $M_a \doteq \frac{1}{2}\epsilon_{abc}M^{bc}$, we obtain the components of a pseudovector which we could call the pseudoclassical total angular momentum. We consider it relevant to comment that the true conserved quantity associated with rotation symmetry is the $(2,0)$-tensor with components M^{ab} and not the total angular momentum itself. On the other hand, in this description, we have obtained the spin angular momentum as a quantity associated with some kind of rotation symmetry in the pseudoclassical spin degrees of freedom.

One aspect that we must not lose sight of is that even though angular momentum is a very relevant physical quantity, in practice there is no way to measure it directly; however, we can measure another closely related quantity, namely, the magnetic moment, associated with a nonstatic charged particle, which is responsible for such a

[k] In our convention $i\epsilon Q = -\delta q p - \delta\theta\pi + \delta t h + i\epsilon\Phi$, with $h = q\cdot p + \theta\cdot\pi - L$ and $\delta L = i\epsilon\frac{d\Phi}{dt}$, being $i\epsilon$ a real parameter.

particle interacting with an external electromagnetic field. It is known from classical electromagnetism that the potential energy associated with the interaction of an electron, describing a circular trajectory, with an external magnetic field $\vec{B}$ is given by $E_{p,L} = -\vec{\mu}_L \cdot \vec{B}$, with $\vec{\mu}_L = -g_L \frac{\mu_B}{\hbar} \vec{L}$, in which μ_B is the Bohr magneton, $g_L = 1$ is the electron g-factor and $\vec{L}$, the angular momentum. Now, if the spin angular momentum $\vec{S}$ is taken into account, an additional term to potential energy arises in the form: $E_{p,S} = -\vec{\mu}_S \cdot \vec{B}$, in which $\vec{\mu}_S = -g_S \frac{\mu_B}{\hbar} \vec{S}$, being g_S the electron spin g-factor.[l] Therefore, the total interaction energy will be given by

$$E = -(\vec{\mu}_L + \vec{\mu}_S) \cdot \vec{B} = \frac{\mu_B}{\hbar}(g_L \vec{L} + g_s \vec{S}) \cdot \vec{B}, \tag{44}$$

from which one concludes that it is not the total angular momentum $\vec{M} = \vec{L} + \vec{S}$ that is present, but the linear combination $g_L \vec{L} + g_s \vec{S}$.

On the other hand, the mechanical description of an electron moving in an external electromagnetic field (characterized by the potentials φ and $\vec{A}$) is described by the well-known Lagrangian:

$$L = \frac{m}{2} \delta_{ab} \dot{x}^a \dot{x}^b - e\dot{x}^a A_a + e\varphi. \tag{45}$$

Nevertheless, this Lagrangian is not complete since it is necessary to include, in addition to the kinematics of spin variables, the contribution due to the interaction of the external magnetic field with the electron spin magnetic moment, namely, the tern $-\vec{\mu}_S \cdot \vec{B}$, which in terms of the θ-variables reads[m]

$$\begin{aligned} -\vec{\mu}_S \cdot \vec{B} &= \frac{e}{m} \delta^{ab} S_a B_b \overset{(41)}{=} -\frac{ie}{2m} \delta^{ab} \epsilon_{acd} \theta^c \theta^d B_b \\ &= -\frac{ie}{2m} \theta^c \theta^d \epsilon^b{}_{cd} B_b = -\frac{ie}{2m} \theta^c \theta^d F_{cd} \end{aligned} \tag{46}$$

with $F_{ab} = \partial_a A_b - \partial_b A_a$. Therefore, to describe completely (i.e. considering also the spin degrees of freedom) the dynamics of an electron in an external electromagnetic field, the use of the following pseudoclassical Lagrangian must be considered:

$$L = L_F + e\left(\varphi - \dot{x}^a A_a + \frac{i}{2m} \theta^a \theta^b F_{ab}\right), \tag{47}$$

in which the first term corresponds to the kinetic part (free case, see Eq. (24)) whereas the second one, is to the interaction sector. This Lagrangian can also be written in terms of the quantity S^{ab} defined in (38):

$$L = L_F + e\left(\varphi - \dot{x}^a A_a - \frac{1}{2m} S^{ab} F_{ab}\right), \tag{48}$$

in which the configurations are represented in the superspace $\mathbb{R}^{3|3}$. Note that this Lagrangian is an improved version of (45). Besides, note the presence of the rank 2 tensor spin angular momentum introduced in (38). Another interesting case in which

[l] Its experimental value is $g_s = 2,00231930436256(35)$, although the Dirac equation provides $g_s = 2$.
[m] For this computation we are considering $g_s = 2$.

we could introduce Grassmann coordinates to describe spin degrees of freedom is the so-called spin–orbit interaction, which takes place in hydrogen-like atoms and is nothing but the interaction between the electron spin and the internal magnetic field of the atom itself.[n] The corresponding energy turns out to be proportional to the product $\vec{S} \cdot \vec{L}$, which written in terms of both ordinary and Grassmann coordinates, according to (41), is given by

$$\vec{S} \cdot \vec{L} = -\frac{i}{2}\theta^a\theta^b(\delta_{ac}x^c p_b - \delta_{bc}x^c p_a) = \frac{1}{2}S^{ab}L_{ab}. \tag{49}$$

Thus, the Lagrangian which correctly describes the spin–orbit interaction is[o]

$$\begin{aligned} L &= \delta_{ab}\left(\frac{m}{2}\dot{x}^a\dot{x}^b - \frac{i}{2}\dot{\theta}^a\theta^b\right) + \frac{iZe^2}{4\pi\varepsilon_0(2mc^2r^3)}\theta_a\theta_b x^a\dot{x}^b \\ &= L_F - \frac{Ze^2}{4\pi\varepsilon_0(4mc^2r^3)}S^{ab}L_{ab}. \end{aligned} \tag{50}$$

At this point, we must observe that the Lagrangians (48) and (50) are both of the form

$$L = \delta_{ab}\left(\frac{m}{2}\dot{x}^a\dot{x}^b - \frac{i}{2}\dot{\theta}^a\theta^b\right) - U - \frac{1}{2}S^{ab}V_{ab}, \tag{51}$$

in which U is a scalar function and V_{ab} denotes the components of an antisymmetric tensor, both depending on the coordinates and possibly linear in velocities. On the other hand, defining $V_a \doteq \frac{1}{2}\epsilon_a{}^{bc}V_{bc}$ (in a (40) fashion) leads to

$$\vec{S} \cdot \vec{V} = \delta^{ab}S_a V_b = \frac{1}{2}S^{ab}V_{ab} \tag{52}$$

and then, (51) can be expressed as

$$L = \frac{m}{2}\dot{\vec{x}}^2 - \frac{i}{2}\dot{\vec{\theta}} \cdot \vec{\theta} - U - \vec{S} \cdot \vec{V}. \tag{53}$$

Hence, either (51) or (53) is the more general Lagrangian describing interactions that involve the electron spin angular momentum. The Lagrangian considered in the case of a nonrelativistic particle in an external electromagnetic field can be put into this form and then the corresponding Hamiltonian is obtained:

$$\begin{cases} L = L_F + e\varphi - e\dot{\vec{x}} \cdot \vec{A} - \dfrac{e}{m}\vec{S} \cdot \vec{B}, \\ H = \dfrac{1}{2m}(\vec{p} + e\vec{A})^2 - e\varphi + \dfrac{e}{m}\vec{S} \cdot \vec{B} \end{cases} \tag{54}$$

[n] In the electron referential frame, the nucleus is orbiting around it. The effective electric current due to such translation produces a magnetic field with the same orientation of the angular momentum of the nucleus (which in turn is identical to that of the electron in the nucleus frame).

[o] Here, we have used the complete expression for the spin–orbit energy, $E_{\mathrm{SO}} = \frac{Ze^2\vec{S}\cdot\vec{L}}{4\pi\epsilon_0(2m^2c^2r^3)}$ and the antisymmetry of $\theta_a\theta_b$.

with $p_a = m\delta_{ab}\dot{x}^b + qA_a$ and using the fact that $B_a = \frac{1}{2}\epsilon_a{}^{bc}F_{bc}$. Note that the last term in H corresponds to the interaction between the spin magnetic moment of the particle and the external field.

3.2. *Case 2: Witten supersymmetric model*

This model was originally introduced by Witten[15] in its quantized version, which led to the so-called supersymmetric quantum mechanics, which have been extensively studied later. The pseudoclassical Lagrangian description for the Witten model considers a configuration superspace $\mathbb{R}^{1|2}$ and the following Lagrangian:

$$L = \frac{m}{2}\dot{x}^2 - \frac{1}{2}W^2 - \frac{i}{2}(\dot{\theta}\theta + \dot{\bar{\theta}}\bar{\theta}) - \frac{i}{\sqrt{m}}\theta\bar{\theta}\frac{dW}{dx}, \tag{55}$$

in which $W = W(x)$ is a Grassmann-even superfunction called *supersymmetric potential*; on the other hand, θ and $\bar{\theta}$ are Grassmann variables. First of all the canonical momenta are obtained:

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \tag{56}$$

$$\bar{\pi} = \frac{\partial_L}{\partial\dot{\theta}}L = -\frac{i}{2}\theta, \quad \pi = \frac{\partial_L}{\partial\dot{\bar{\theta}}}L = -\frac{i}{2}\bar{\theta}. \tag{57}$$

Then, the Hamiltonian reads

$$\begin{aligned} H &= \dot{x}p + \dot{\theta}\bar{\pi} + \dot{\bar{\theta}}\pi - L \\ &= \frac{1}{2m}p^2 + \frac{1}{2}W^2 + \frac{i}{\sqrt{m}}\theta\bar{\theta}\frac{dW}{dx}. \end{aligned} \tag{58}$$

Due to the dependence between the odd momentum and coordinate variables, the following coordinates are considered for Faddeev–Jackiw phase superspace $\mathbb{R}^{2|2}$:

$$z^{st} = (x \quad p \quad \theta \quad \bar{\theta}) \tag{59}$$

and so, the canonical Lagrangian is

$$L = \dot{x}p - \frac{i}{2}\dot{\theta}\theta - \frac{i}{2}\dot{\bar{\theta}}\bar{\theta} - H, \tag{60}$$

from it, the canonical 1-form is derived:

$$\vartheta = dxp - \frac{i}{2}d\theta\theta - \frac{i}{2}d\bar{\theta}\bar{\theta}. \tag{61}$$

Thus, by using (11) one gets the following 2-form:

$$\omega = dx \wedge dp + \frac{i}{2}d\theta \wedge d\theta + \frac{i}{2}d\bar{\theta} \wedge d\bar{\theta}, \tag{62}$$

whose supermatrix representation is

$$\left(\begin{array}{cc:cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{array}\right). \tag{63}$$

It is easy to check that the supermatrix above is nonsingular; then, its inverse is presented as follows:

$$\left(\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{array}\right). \tag{64}$$

Therefore, from (23) the GPB of phase superspace variables is obtained:

$$\{x; p\} = 1, \quad \{\theta; \theta\} = -i = \{\bar{\theta}; \bar{\theta}\}. \tag{65}$$

It is interesting to observe that the dynamics generated by (55) remains invariant under the following infinitesimal transformations:

$$x \to x' = x + \delta_j x, \quad \begin{array}{l} \theta \to \theta' = \theta + \delta_j \theta, \\ \bar{\theta} \to \bar{\theta}' = \bar{\theta} + \delta_j \bar{\theta} \end{array} \tag{66}$$

with the following small variations:

$$\begin{cases} \delta_1 x = -i\epsilon_1 \bar{\theta}/\sqrt{m}, \\ \delta_1 \theta = \epsilon_1 W, \\ \delta_1 \bar{\theta} = \epsilon_1 \sqrt{m}\dot{x}, \end{cases} \quad \begin{cases} \delta_2 x = -i\epsilon_2 \theta/\sqrt{m}, \\ \delta_2 \theta = \epsilon_2 \sqrt{m}\dot{x}, \\ \delta_2 \bar{\theta} = -\epsilon_2 W, \end{cases} \tag{67}$$

in which the ϵ_j denotes real parameters with odd Grassmann-parity. The index j states that there are two sets of transformations that determine a symmetry. We must emphasize that these transformations mix even and odd variables and for this reason are commonly known as supersymmetry transformations. It is direct to check that under such transformations the Lagrangian presents the following variations:

$$\begin{cases} \delta_1 L = -\dfrac{i}{2}\epsilon_1 \dfrac{d}{dt}\left(\dfrac{1}{\sqrt{m}}\bar{\theta} p - \theta W\right), \\ \delta_2 L = -\dfrac{i}{2}\epsilon_2 \dfrac{d}{dt}\left(\dfrac{1}{\sqrt{m}}\theta p + \bar{\theta} W\right), \end{cases} \tag{68}$$

which justifies our previous assertion that (66) are supersymmetry transformations. According to the Noether theorem, there must exist a conserved charge associated with each supersymmetry transformation:

$$Q_1 = \frac{1}{\sqrt{m}}\bar{\theta} p + \theta W, \quad Q_2 = \frac{1}{\sqrt{m}}\theta p - \bar{\theta} W. \tag{69}$$

Note that $|Q_j| = 1$; i.e. the Noether charges are anticommutative objects. The Q_j is known in the literature as supercharges and as every Noether charge, it plays the role of generators of the small variations presented in (67):

$$\delta_j z^A = i\epsilon\{Q_j; z^A\}. \tag{70}$$

On the other hand, these supercharges verify a very particular relation:

$$\{Q_j; Q_k\} = -2i\delta_{jk}H \quad (j, k = 1, 2). \tag{71}$$

In particular, the choice $\frac{d}{dx}W = \sqrt{m}\omega$ (with ω constant) yields the following Lagrangian:

$$L = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 - \frac{i}{2}(\dot{\theta}\theta + \dot{\bar{\theta}}\bar{\theta}) - i\omega\theta\bar{\theta}, \tag{72}$$

which in turn leads to the Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 + i\theta\bar{\theta}\omega = \omega\left[\frac{1}{2}(X^2 + P^2) + i\theta\bar{\theta}\right] \tag{73}$$

with $X := (m\omega)^{1/2}x$ and $P := (m\omega)^{-1/2}p$. Readily, we recognize that the sector of (73) involving only ordinary variables corresponds to the well-known harmonic oscillator, in which defining the auxiliary quantities:

$$a \doteq \frac{1}{\sqrt{2}}(X + iP), \quad a^* \doteq \frac{1}{\sqrt{2}}(X - iP), \tag{74}$$

leads to the following nonvanishing classical brackets relations:

$$\{a; H\} = -i\omega a, \quad \{a^*; H\} = i\omega a^*, \quad \{a; a^*\} = -i. \tag{75}$$

Besides that, by using $\dot{F} = \{F; H\}$, one can obtain the laws of motion for the variables introduced in (74):

$$\begin{cases} a = a_0 e^{-i\omega t} \\ a^* = a_0^* e^{i\omega t} \end{cases} \Rightarrow X = \frac{1}{\sqrt{2}}(a_0 e^{-i\omega t} + a_0^* e^{i\omega t}), \tag{76}$$

which exhibits the harmonic behavior of the coordinate x.

In a completely analogous way, one could define[p]

$$b \doteq \frac{1}{\sqrt{2}}(\theta + i\bar{\theta}), \quad b^* \doteq \frac{1}{\sqrt{2}}(\theta - i\bar{\theta}); \tag{77}$$

which turn out to be such that

$$\{b; H\} = -i\omega b, \quad \{b^*; H\} = i\omega b^*, \quad \{b; b^*\} = -i \tag{78}$$

[p] A quick calculation shows that the sector of the Lagrangian (72) involving only odd coordinates can be written in terms of the complex odd variables defined in (77) as $L = -\frac{i}{2}(\theta \cdot \theta + \bar{\theta} \cdot \bar{\theta}) - i\omega\theta\bar{\theta} = ib^*\dot{b} + \frac{d}{dt}(b^*b) - \omega b^*b \cong ib^*\dot{b} - \omega b^*b$.

and as a consequence:

$$\begin{cases} \theta = \dfrac{1}{\sqrt{2}}(b_0 e^{-i\omega t} + b_0^* e^{i\omega t}), \\ \bar{\theta} = \dfrac{1}{\sqrt{2}i}(b_0 e^{-i\omega t} - b_0^* e^{i\omega t}), \end{cases} \tag{79}$$

which reveals that the Grassmann variables that describe this system also have an oscillatory behavior. Thus, since all of the configuration variables describe an oscillatory motion with (angular) frequency ω, this particular case of the Witten model is known as pseudoclassical supersymmetric harmonic oscillator.

Another interesting pseudoclassical Lagrangian that exhibits this kind of supersymmetry is the one presented in (48) in the particular case in which $\varphi = 0$. Then, for a spin-1/2 particle with electric charge $-e$ and spin g-factor that can be rounded to 2 (e.g. electron and muon), the Lagrangian becomes

$$L = L_F - e\left(\dot{x}^a A_a + \frac{1}{2m} S^{ab} F_{ab}\right). \tag{80}$$

In this case, the small variations,

$$\delta x^a = -i\frac{\epsilon}{\sqrt{m}}\theta^a, \quad \delta\theta^a = \epsilon\sqrt{m}\dot{x}^a, \tag{81}$$

induce the variation in the Lagrangian:

$$\delta L = i\epsilon\frac{d}{dt}\left(-\frac{\sqrt{m}}{2}\delta_{ab}\dot{x}^a\theta^b + \frac{e}{\sqrt{m}}\theta^a A_a\right), \tag{82}$$

which shows that, in fact, this is a supersymmetry. Besides, the corresponding supercharge results:

$$Q = \frac{1}{\sqrt{m}}\theta^a(p_a + eA_a). \tag{83}$$

Then, the relation in (71) yields the following Hamiltonian:

$$H = \frac{1}{2m}(p_a + eA_a)\delta^{ab}(p_b + eA_b) + \frac{e}{2m}S^{ab}F_{ab} \tag{84}$$

or in vector notation:

$$H = \frac{1}{2m}(\vec{p} + e\vec{A})^2 + \frac{e}{m}\vec{S}\cdot\vec{B}, \tag{85}$$

which, of course, coincides with the one obtained through the pseudomechanical canonical construction (see Eq. (54)). Note that the transformations corresponding to (81) do not depend on the vector potential $\vec{A}$ and this suggests that the system exhibits supersymmetry under such transformations, even in the noninteracting case, described by the Lagrangian (24), with supercharge $Q = \frac{1}{\sqrt{m}}p_a\theta^a$.

4. Canonical Quantization

Among the different known quantization schemes, probably the most widely used is canonical quantization, which consists of establishing a correspondence between the classical Poisson brackets of canonical variables and the commutation (anticommutation) relations of the associated quantum operators via the correspondence principle. Explicitly, one uses the recipe:

$$\{F; G\} \to \frac{1}{i\hbar}[\hat{F}; \hat{G}]_{\pm}, \tag{86}$$

in which the square brackets $[\cdot;\cdot]_-$ ($[\cdot;\cdot]_+$) stand for commutator (anticommutator) when both dynamical functions have equal Grassmann-parity $|F| = 0 = |G|$ ($|F| = 1 = |G|$). More generally, the graded classical brackets become the superbrackets:

$$[\hat{F}; \hat{G}]_{\pm} = \hat{F}\hat{G} - (-1)^{|F||G|}\hat{G}\hat{F}, \tag{87}$$

in which the operators act on a Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1$. Technically, a quantization of a pseudoclassical system consists on finding a representation of the Lie superalgebra of dynamical superfunctions, with the classical superbrackets given by (65) for a phase superspace $\mathbb{R}^{2|2}$, or (35) for a general phase superspace $\mathbb{R}^{2m|n\geq 2}$, playing the role of Lie superbrackets.[16]

In the remaining part of this section, the canonical quantization of the cases presented in the previous section will be performed, focusing on the Grassmann-odd variables, since the even ones are already known.

4.1. *Case 1: Pseudoclassical spin degrees of freedom*

The starting point is to apply the rule (86) to the classical brackets obtained in (35):

$$\{\theta^a; \theta^b\} = -i\delta^{ab} \to [\hat{\theta}^a; \hat{\theta}^b]_+ = \hbar\delta^{ab} \quad (a, b = 1, 2, 3), \tag{88}$$

from them, one easily recognizes that these anticommutation relations correspond to a Clifford algebra $\mathcal{C}\ell(3, \mathbb{C})$ according to

$$\left[\sqrt{\frac{2}{\hbar}}\hat{\theta}^a; \sqrt{\frac{2}{\hbar}}\hat{\theta}^b\right]_+ = 2\delta^{ab} \tag{89}$$

and taking into account that this algebra is generated by the Pauli matrices σ_a, which obey an identical anticommutation relation: $[\sigma_a; \sigma_b]_+ = 2\delta_{ab}I$, one may choose the following representation:

$$\hat{\theta}^a = \sqrt{\frac{\hbar}{2}}\delta^{ab}\sigma_b, \tag{90}$$

which implies that the quantum operators corresponding to the spin degrees of freedom are also self-adjoint. With this, the quantized version of the even variables

S_a defined in (41) acquires the form

$$\hat{S}_a = -\frac{i}{2}\epsilon_{abc}\left(\sqrt{\frac{\hbar}{2}}\delta^{bm}\sigma_m\right)\left(\sqrt{\frac{\hbar}{2}}\delta^{cn}\sigma_n\right) = \frac{\hbar}{2}\sigma_a. \tag{91}$$

Besides, applying the canonical quantization procedure, the classical brackets between the quantities presented in (43) become

$$\{S_a; S_b\} = \epsilon_{ab}{}^{c} S_c \to [\hat{S}_a; \hat{S}_b]_- = i\hbar\epsilon_{ab}{}^{c}\hat{S}_c, \tag{92}$$

which is the expected result, since it corresponds to the quantum commutation relation of an angular momentum operator. More rigorously, since the quantum Hilbert space for this system is $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, one should write $\hat{S}_a = \mathbf{1} \otimes \frac{\hbar}{2}\sigma_a$. On the other hand, considering the case in which the particle is charged in a region with an external magnetic field, the Hamiltonian in (54) becomes, after quantization

$$\hat{H} = \left[\frac{1}{2m}(\underline{\hat{p}} + e\underline{\hat{A}})^2 - e\hat{\varphi}\right] \otimes \mathbf{1} + \frac{e\hbar}{2m}\underline{\hat{B}} \cdot \vec{\sigma} \tag{93}$$

with

$$\underline{\hat{A}} = \vec{A}(\underline{\hat{x}}), \quad \hat{\varphi} = \varphi(\underline{\hat{x}}), \quad \underline{\hat{B}} = \vec{B}(\underline{\hat{x}}). \tag{94}$$

We immediately recognize that (93) corresponds to the well-known Pauli Hamiltonian.

4.2. *Case 2: Witten supersymmetric model*

For the sake of perspicuity, we start the quantization program for the particular case of supersymmetric harmonic oscillator; later it will be extended to the general case. The starting point consists in obtaining the quantum brackets corresponding to (65):

$$\{\theta; \theta\} = -i = \{\bar{\theta}; \bar{\theta}\} \to [\hat{\theta}; \hat{\theta}]_+ = \hbar = [\hat{\bar{\theta}}; \hat{\bar{\theta}}]_+. \tag{95}$$

Besides that, quantizing the classical brackets in (78) yields[q]

$$[\hat{b}; \hat{b}^\dagger]_+ = \hbar, \tag{96}$$

from these anticommutation relations, one identifies the underlying Clifford algebra $\mathcal{C}\ell(2, \mathbb{C})$ according to

$$\left[\frac{1}{\sqrt{\hbar}}\hat{b}; \frac{1}{\sqrt{\hbar}}\hat{b}^\dagger\right]_+ = 1. \tag{97}$$

Therefore, taking into account that the matrices $\sigma_\pm \doteq \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ satisfy the anticommutation relation: $[\sigma_+; \sigma_-]_+ = I$, we may choose the following representation:

$$\hat{b} = \sqrt{\hbar}\sigma_+, \quad \hat{b}^\dagger = \sqrt{\hbar}\sigma_-, \tag{98}$$

[q] According (77) and assuming that $\hat{\theta}$, $\hat{\bar{\theta}}$ are self-adjoint (motivated by the previous example), it turns out that $\widehat{b^*} = \hat{b}^\dagger$.

which in turn implies

$$\hat{\theta} = \sqrt{\frac{\hbar}{2}}\sigma_1, \quad \hat{\bar{\theta}} = \sqrt{\frac{\hbar}{2}}\sigma_2. \tag{99}$$

Considering this result, the corresponding quantum Hamiltonian, acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$ reads

$$\hat{H} = \left(\frac{\hat{p}^2}{2m} + \frac{m}{2}\omega^2\hat{x}^2\right) \otimes \mathbf{1} - \mathbf{1} \otimes \frac{\hbar}{2}\omega\sigma_3, \tag{100}$$

which is of the form $\hat{H} = \hat{H}_B \otimes \mathbf{1} + \mathbf{1} \otimes \hat{H}_F$. The sector $\hat{H}_B : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ ($\hat{H}_F : \mathbb{C}^2 \to \mathbb{C}^2$) of this Hamiltonian corresponds to the so-called *bosonic* (*fermionic*) quantum harmonic oscillator. Maintaining the representation (99) for the θ-quantum operators, we arrive at the following expression for the quantum Hamiltonian of the Witten model:

$$\hat{H} = \left(\frac{\hat{p}^2}{2m} + \frac{1}{2}\hat{\mathrm{W}}^2\right) \otimes \mathbf{1} - \mathbf{1} \otimes \frac{\hbar}{2\sqrt{m}}\widehat{W'}\sigma_3 \tag{101}$$

with

$$\hat{\mathrm{W}} = W(\hat{x}), \quad \widehat{W'} = \frac{dW}{dx}(\hat{x}). \tag{102}$$

4.3. *Supersymmetric quantum mechanics*

A quantum system characterized by a Hamiltonian $\hat{H}$ and a set of self-adjoint operators $\{\hat{Q}_j\}_{j=1}^N$, with all of them acting on some Hilbert space, is said to be *supersymmetric* if

$$[\hat{Q}_j; \hat{Q}_k]_+ = \delta_{jk}\hat{H}; \quad \forall, 1 \leq j, k \leq N. \tag{103}$$

In this context, the operators $\hat{Q}_j$ are called quantum supercharges and we say that the system presents N-extended supersymmetry.

Applying the canonical quantization procedure to (71) results in

$$[\hat{Q}_j; \hat{Q}_k]_+ = 2\hbar\delta_{jk}\hat{H} \tag{104}$$

and then, we see the quantized Noether charges due to the (pseudoclassical) supersymmetry transformations coincide with the quantum supercharges up to a scale factor $(2\hbar)^{-1/2}$. Then, for example, the quantum Witten model turns out to be a system with $N = 2$ supersymmetry and supercharge operators:

$$\begin{cases} \hat{Q}_1 = \dfrac{1}{2}\left(\dfrac{1}{\sqrt{m}}\hat{p} \otimes \sigma_2 + \hat{\mathrm{W}} \otimes \sigma_1\right), \\ \hat{Q}_2 = \dfrac{1}{2}\left(\dfrac{1}{\sqrt{m}}\hat{p} \otimes \sigma_1 - \hat{\mathrm{W}} \otimes \sigma_2\right), \end{cases} \tag{105}$$

which was obtained from (69) and (99). Another quantum system with $N = 1$ supersymmetry is the one characterized by the Hamiltonian (93), which corresponds to a charged spin-1/2 particle in an external magnetic field. The associated quantum supercharge is given by

$$\hat{Q} = \frac{1}{2\sqrt{m}}\,(\underline{\hat{p}} + e\underline{\hat{A}}) \cdot \vec{\sigma}, \tag{106}$$

which was obtained from (83) and (90).

5. Conclusions

The Faddeev–Jackiw formalism was developed for discrete systems described by both commutative and anticommutative variables. The construction was made on a set of supercoordinates that we call phase superspace, where the Lagrangian was written in its canonical form. The corresponding equations of motion were consistently derived from Hamilton's principle, exhibiting a natural symplectic structure, which has several relevant properties, highlighting among them the fact that its inverse determines the graded Poisson brackets (GPB). The fundamental GPB for the supercoordinates z^A was determined from the components of the inverse of the symplectic form, and with them, the GPB of any pair of dynamical superfunctions can be easily obtained.

The case of pseudoclassical spin degrees of freedom was carefully studied and suitable initial conditions were introduced to solve the variational problem. The (nonsingular) supermatrix representation of the symplectic form was obtained, and its inverse allowed us to determine the nonnull fundamental GPB of the model. Noether charge corresponding to (rigid) $O(3)$ symmetry was derived leading to the total angular momentum straightforwardly. Finally, the general Lagrangian describing interactions that involve the electron spin angular momentum was deduced.

The Witten model was treated at a pseudoclassical level and it was shown to be invariant under supersymmetry transformations. The corresponding Noether charges act as generators of these transformations. Interestingly, from the fundamental GPB of the model, Noether charges were found to satisfy a very particular *algebra* involving the Hamiltonian itself. It was also verified that this model includes the supersymmetric harmonic oscillator as a particularly interesting case.

The canonical quantization program was developed for the studied models, leading to the conclusion that the quantum operators corresponding to the Grassmann-odd degrees of freedom satisfy a complex Clifford algebra, which implies that their proper representation must be given by Pauli matrices (or linear combinations of them). In addition, we found that the Hamiltonian obtained by the pseudoclassical Lagrangian proposed for electromagnetic interactions involving spin degrees of freedom becomes the Pauli Hamiltonian after being quantized. On the other hand, it was seen that the purely magnetic interactions described by this Lagrangian also display supersymmetric behavior.

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References

1. S. I. Tomonaga, *The Story of Spin* (University of Chicago Press, 1997).
2. R. Casalbuoni, *Nuov. Cim. A* **33**, 115 (1976).
3. R. Casalbuoni, *Nuov. Cim. A* **33**, 389 (1976).
4. F. A. Berezin and M. S. Marinov, *Pis'ma Zh. Eksp. Teor. Fiz.* **21**, 678 (1975).
5. F. A. Berezin and M. S. Marinov, *Ann. Phys.* **104**, 336 (1977).
6. P. A. M. Dirac, *Proc. R. Soc. A* **246**, 326 (1958).
7. L. Faddeev and R. Jackiw, *Phys. Rev. Lett.* **60**, 1692 (1988).
8. J. Govaerts, *Int. J. Mod. Phys. A* **5**, 3625 (1990).
9. E. C. Manavella, *Int. J. Mod. Phys. A* **27**, 1250145 (2012).
10. D. V. Soroka and V. A. Soroka, *J. High Energy Phys.* **2003**, 001 (2003).
11. T. Covolo and N. Poncin, Lectures on Supergeometry (2012).
12. P. Deligne and J. Morgan, Notes on supersymmetry (following Joseph Bernstein), in *Quantum Fields and Strings: A Course for Mathematicians* (American Mathematical Society, Providence, RI, 1999), pp. 41–97.
13. L. G. Caro, G. B. de Gracia, A. A. Nogueira and B. M. Pimentel, *Nucl. Phys. B* **994**, 116311 (2023).
14. C. A. Galvão and C. Teitelboim, *J. Math. Phys.* **21**, 1863 (1980).
15. E. Witten, *Nucl. Phys. B* **188**, 513 (1981).
16. P. Woit, *Quantum Theory, Groups and Representations* (Springer International Publishing, 2017).