Gauge Symmetry in the Hamilton-Jacobi Formulation

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Proca Field with Gauge Symmetry

The Proca field is described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The gauge symmetry can be incorporated by adding a nonlocal and non-polynomial term to (1). If the mass term transforms as

$$\frac{1}{2}m^2 A_\mu A^\mu \to \frac{1}{2}m^2 \left(A_\mu + \partial_\mu \Lambda\right)\left(A^\mu + \partial^\mu \Lambda\right), \tag{2}$$

and the parameter Λ is redefined to rewrite the mass term as

$$\frac{1}{2}m^2 \left[A_\mu + \frac{1}{e}\partial_\mu \theta\right]^2. \tag{3}$$

This expression becomes gauge invariant under

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x), \qquad \theta(x) \to \theta(x) - e\Lambda(x), \tag{4}$$

with $\Box\theta \neq 0$, where θ is an auxiliary scalar field and e a coupling constant. This leads to the effective gauge-invariant Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 \left[A_\mu + \frac{1}{e}\partial_\mu\theta\right]^2 . \tag{5}$$

From (5), the corresponding Euler–Lagrange equations follow:

$$\partial_\nu F^{\nu\mu} + m^2 A^\mu = -\frac{1}{e}m^2\partial^\mu\theta, \qquad \partial_\nu \left[A^\nu + \frac{1}{e}\partial^\nu\theta\right] = 0. \tag{6}$$

Hamilton-Jacobi Formulation

The canonical momenta associated to the filds A_μ and θ are:

$$\pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu} \quad \text{y} \quad p_\theta \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \theta)} = \frac{m^2}{e}\left(A_0 + \frac{1}{e}\partial_0\theta\right). \tag{7}$$

From the definition of the canonical momenta associated to the filed A_μ we obtain

$$\phi^1 \equiv \pi^0 = 0\,. \tag{8}$$

The Hamilton-Jacobi (HJ) equation is definded by,

$$\frac{\partial S}{\partial t} + H_C = 0, \tag{9}$$

The Hamiltonian is defined by:

$$H_C = \int d^3x \mathcal{H}_C. \tag{10}$$

where the canonical Hamiltonian density is:

$$\mathcal{H}_C = \pi^{\mu}(x)\,\dot{A}_{\mu}(x) + p_{\theta}(x)\,\dot{\theta}(x) - \mathcal{L}. \tag{11}$$

and it takes the form

$$\begin{aligned} \mathcal{H}_C &= \frac{1}{2}\pi^k\pi^k + \frac{1}{2}\frac{e^2}{m^2}p_\theta^2 + \pi^k\partial_k A_0 - eA_0 p_\theta \\ &+\frac{1}{4}F_{ki}F^{ki} + \frac{m^2}{2}\left(A_k + \frac{1}{e}\partial_k\theta\right)^2, \end{aligned} \tag{12}$$

We have the initial set of Hamilton-Jacobi Partial Differential Equations (HJPDE)

$$\phi^0 \equiv p^t + \mathcal{H}_c = 0, \qquad \rightarrow \qquad t, \tag{13}$$

$$\phi^1 \equiv \pi^0 = 0, \qquad \rightarrow \qquad \omega_1. \tag{14}$$

Let us define $p^t \equiv \partial_0 S$. The phase is characterized by the variables: A_μ, π^μ, θ and p_θ. The fundamental Poisson Brackets (PB) are defined by

$$\begin{aligned} \{F(x), G(y)\}_{x_0=y_0} \equiv & \int d^3z \left[\frac{\delta F(x)}{\delta A_\mu(z)}\frac{\delta G(y)}{\delta \pi^\mu(z)} - \frac{\delta F(x)}{\delta \pi^\mu(z)}\frac{\delta G(y)}{\delta A_\mu(z)}\right] \\ & + \int d^3z \left[\frac{\delta F(x)}{\delta \theta(z)}\frac{\delta G(y)}{\delta p_\theta(z)} - \frac{\delta F(x)}{\delta p_\theta(z)}\frac{\delta G(y)}{\delta \theta(z)}\right]. \end{aligned}$$

The non null fundamental Poisson brackets are given by

$$\begin{aligned} \left\{A_{\mu}(x), \pi^{\nu}(y)\right\} &= \delta_{\mu}^{\nu}\delta^{3}(x-y), \\ \left\{\theta(x), p_{\theta}(y)\right\} &= \delta^{3}(x-y). \end{aligned} \quad (15)$$

The dynamic evolution of any function defined in the phase space is given by,

$$dF(x) = \int d^{3}y \left[\left\{F(x), \phi^{0}(y)\right\} dt + \left\{F(x), \phi^{1}(y)\right\} d\omega_{1}(y)\right]. \quad (16)$$

Integrability Condition (IC)

The IC on the equations ϕ^1 imply the following relation

$$d\phi^1(x) = \int d^3y \left[\left\{ \phi^1(x), \phi^0(y) \right\} dt + \left\{ \phi^1(x), \phi^1(y) \right\} d\omega_1(y) \right] = 0. \tag{17}$$

be imposed on the system. The condition for ϕ^1 gives,

$$d\phi^1 = \left[\partial_k \pi^k + e p_\theta \right] dt = 0. \tag{18}$$

Since ϕ^1 is not in involution, IC give new HJPDE:

$$\phi^2 \equiv \partial_k^x \pi^k(x) + e p_\theta(x) = 0, \tag{19}$$

and the IC have to be tested with this too.

It is possible to show that IC is automatically satisfied, i.e,

$$d\phi^2 = 0. \tag{20}$$

Thus, the system ϕ^1 and ϕ^2 is completely integrable. The final algebra between the HJPDE is given by

$$\left\{\phi^i(x), \phi^j(y)\right\} = 0 \qquad \text{with} \qquad i = 1, 2. \tag{21}$$

This relation indicate that the set ϕ^0, ϕ^1 and ϕ^2 are involutive with the PBs, thus, the integrability is achieved.

Characteristic Equations (CE)

With the complete set of HJPDE: ϕ^0, $\mathcal{G}^\lambda \equiv \phi^1$ and $\mathcal{G}^\omega \equiv \phi^2$, the fundamental differential is now given by:

$$dF(x) = \int d^2y \Big[\left\{F(x), \phi^{\,0}(y)\right\}^* dx^0 + \left\{F(x), \mathcal{G}^\lambda(y)\right\}^* d\lambda(y) + \left\{F(x), \mathcal{G}^\omega(y)\right\}^* d\omega(y)\Big]. \tag{22}$$

From where we obtain the characteristic equations

$$\begin{aligned} dA_\mu &= \delta^k_\mu\left(\pi^k + \partial_k A_0\right)dt + \delta^0_\mu d\lambda - \delta^k_\mu \partial_k d\omega, \\ d\pi^\mu(x) &= \left\{\delta^\mu_0\left(\partial_k\pi^k + ep_\theta\right) + \delta^\mu_k\left[\partial_l F^{kl} - m^2\left(A_k + \frac{1}{e}\partial_k\theta\right)\right]\right\}dt, \\ d\theta(x) &= \left[\frac{e^2}{m^2}p_\theta - eA_0\right]dt + ed\omega, \\ dp_\theta(x) &= \frac{m^2}{e}\partial_k\left(A_k + \frac{1}{e}\partial_k\theta\right)dt. \end{aligned} \tag{23}$$

The dynamic evolution of the system depends on the parameters t, λ and ω. The IC states that these parameters are independents.

If we consider the evolution in the temporal direction, we obtain

$$\begin{aligned}
\partial_0 A_\mu &= \delta_\mu^k \left(\pi^k + \partial_k A_0\right), \\
\partial_0 \pi^\mu (x) &= \delta_0^\mu \left(\partial_k \pi^k + e p_\theta\right) + \delta_k^\mu \left[\partial_l F^{kl} - m^2 \left(A_k + \frac{1}{e}\partial_k \theta\right)\right], \\
\partial_0 \theta (x) &= \left[\frac{e^2}{m^2} p_\theta - e A_0\right], \\
\partial_0 p_\theta (x) &= \frac{m^2}{e}\partial_k \left(A_k + \frac{1}{e}\partial_k \theta\right).
\end{aligned} \tag{24}$$

Gauge Symmetry

If the parameters λ and ω depend on time, then the dynamics of the fields are given by

$$\begin{aligned}
\dot{A}_\mu &= \delta^0_\mu \dot{\lambda} + \delta^k_\mu \left(\pi^k - \partial_k A_0\right) + \delta^k_\mu \partial_k \dot{\omega}, \\
\dot{\pi}^\mu &= \delta^\mu_0 \left(\partial_k \pi^k + e p_\theta\right) + \delta^\mu_k \left[\partial_l F^{lk} - m^2 \left(A_k + \frac{1}{e}\partial_k \theta\right)\right], \\
\dot{\theta} &= \frac{e^2}{m^2} p_\theta - e A_0 + e\dot{\omega} \ , \\
\dot{p}_\theta &= \frac{m^2}{e} \partial_k \left(A_k + \frac{1}{e}\partial_k \theta\right).
\end{aligned} \tag{25}$$

These equations can be combined and result in

$$\partial_\mu F^{\mu\nu} + m^2 \left[A^\nu + \tfrac{1}{e}\partial^\nu \theta\right] = 0 \quad , \quad \partial_\mu \left[A^\mu + \tfrac{1}{e}\partial^\mu \theta\right] = \partial_0 \dot{\omega}. \tag{26}$$

The evolution along the parameters λ and ω are identified as variations denoted with δ_λ and δ_ω they are given by

$$\begin{aligned} \delta_\lambda A_\mu &= \delta^0_\mu \delta\lambda, \\ \delta_\omega A_\mu &= -\delta^k_\mu \partial_k \delta\omega, \\ \delta_\omega \theta &= e\delta\omega. \end{aligned} \tag{27}$$

In order to consider local transformations we must consider that this parameters λ and ω are now depend on the time variable. Thus,

$$\begin{aligned} \delta A_\mu &= \delta^0_\mu \delta\lambda - \delta^k_\mu \partial_k \delta\omega, \\ \delta\theta &= e\delta\omega. \end{aligned} \tag{28}$$

The set of local variations (28) is now generated by the linear combinations of the involutive Hamiltonians, i.e.,

$$G^{\text{can}} \equiv \int d^2y \left[\mathcal{G}^{\lambda}\delta\lambda + \mathcal{G}^{\omega}\delta\omega\right]. \tag{29}$$

we call to G^{can} the generator of canonical transformations since we have that,

$$\begin{aligned} \delta A_\mu &= \{A_\mu, G^{\text{can}}\} = \delta^0_\mu \delta\lambda - \delta^k_\mu \partial_k \delta\omega, \\ \delta\theta &= \{A_\mu, G^{\text{can}}\} = e\delta\omega. \end{aligned} \tag{30}$$

The variation of the action functional for fixed point variations is,

$$\delta\mathcal{L} = \left[\partial_l F^{l0} + m^2 \left(A^0 + \frac{1}{e}\partial^0\theta\right)\right](\delta\lambda + \partial_0\delta\omega). \tag{31}$$

If the theory is invariant, i.e., $\delta\mathcal{L} = 0$, we should consider

$$\delta\lambda = -\partial_0 \delta\omega. \tag{32}$$

We have one arbitrary parameter, say $\Lambda \equiv \delta\omega$. Under the condition, the transformation on the fields A_μ, θ takes the form

$$\begin{aligned} \delta A_\mu &= \{A_\mu, G^{\text{can}}\} = -\partial_\mu \Lambda. \\ \delta\theta &= \{A_\mu, G^{\text{can}}\} = e\Lambda. \end{aligned} \tag{33}$$

From this condition, the generator of the gauge transformation is

$$G^{\text{can}} = \int d^3y \left(-\mathcal{G}^\lambda \partial_0 + \mathcal{G}^\omega \right) \Lambda. \tag{34}$$

Gauge Conditions and Generalized Brackets

- The existence of involutive constraints: $\mathcal{G}^{\lambda} \equiv \phi^1$ and $\mathcal{G}^{\omega} \equiv \phi^2$, does not allow the dynamics to be uniquely determined.
- The gauge freedom is eliminated by introducing gauge conditions.

$$\phi^3 \equiv A_0 = 0 \qquad , \qquad \phi^4 \equiv \partial_k A_k + \frac{m^2}{e}\theta = 0. \tag{35}$$

The full set of HJPDES is:

$$\begin{aligned}
\phi^0 &\equiv p^t + \mathcal{H}_c = 0,\\
\phi^1 &\equiv \pi^0 = 0,\\
\phi^2 &\equiv \partial_k \pi^k + e p_\theta = 0,\\
\phi^3 &\equiv A_0 = 0,\\
\phi^4 &\equiv \partial_k A_k + \frac{m^2}{e}\theta = 0.
\end{aligned} \tag{36}$$

The dynamics is given by the following fundamental differential:

$$\begin{aligned} dF(x) &= \int d^3y \Big[\left\{F(x), \phi^0(y)\right\} dt + \left\{F(x), \phi^1(y)\right\} d\omega_1(y) \\ &\quad + \left\{F(x), \phi^2(y)\right\} d\omega_2(y) + \left\{F(x), \phi^3(y)\right\} d\omega_3(y) \\ &\quad + \left\{F(x), \phi^4(y)\right\} d\omega_4(y) \Big]. \end{aligned} \tag{37}$$

By introducing the gauge conditions the set of parameters $(\omega_1, \omega_2, \omega_3, \omega_4)$ becomes dependent on the parameter t, namely

$$d\omega_i(y) = -\iint d^2u d^2v \Phi_{ij}^{-1}(y, v) \left\{\phi^j(v), \phi^0(u)\right\} dt. \tag{38}$$

Where the matrix $\Phi_{ij}(\mathbf{x},\mathbf{y}) \equiv \left\{\phi^i(x), \phi^j(y)\right\}$ and its inverse given by

$$\Phi^{-1}(x,y) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{D_x} \\ -1 & 0 & 0 & 0 \\ 0 & \frac{1}{D_x} & 0 & 0 \end{pmatrix} \delta^3(x-y)\,, \qquad (39)$$

with $D_x \equiv \nabla_x - m^2$.

Thus, the fundamental differential can be written as

$$dF(x) = \int d^3y \left\{ F(x), \phi^0(y) \right\}^* dt. \qquad (40)$$

The expression $\{\,,\}^*$ identifies what is called of Generalized Brackets (GB), which is defined by

$$\{F(x), G(y)\}^* \equiv \{F(x), G(y)\} - \int\int d^3u d^3v \{F(x), \phi^n(u)\} \left(\Phi^{-1}\right)^{nl}(u,v) \left\{\phi^l(v), G(y)\right\}. \quad (41)$$

The GB satisfy the following property

$$\left\{F(x), \phi^i(y)\right\}^* = 0, \qquad i = 1,2,3,4. \quad (42)$$

The degrees of freedom of the system are (A_i, π^i) and their GB are

$$\left\{A_i\left(x\right), \pi^k\left(y\right)\right\}^* = \left(\delta_i^k + \frac{\partial_i^x \partial_k^x}{D_x}\right)\delta^3\left(x-y\right). \tag{43}$$

Conclusions

- We have calculated the complete set of HJPDES.
- We show that the HJPDES are involutive, which characterizes the system as a gauge theory.
- We calculate the condition for the canonical transformation generator to be a gauge transformation.
- Gauge conditions were introduced in order to eliminate the gauge freedom of the theory,
- The GC were calculated.