Hamilton-Jacobi Formulation of the Proca Field with Guage Symmetry

G. E. R. Zambrano and J. F. Portillo

Departamento de Física,
Universidad de Nariño,
San Juan de Pasto, Nariño, Colombia

Abstract

We analyze the constraint structure of the Proca field with gauge symmetry within the framework of the Hamilton-Jacobi formalism. The complete set of Hamiltonians generating the systems dynamics is derived from Frobenius integrability conditions, together with the corresponding characteristic equations. As generators of canonical transformations, the Hamiltonians are naturally related to the generators of the Lagrangian gauge transformations. Finally, suitable gauge conditions are imposed, and the generalized brackets are explicitly determined.

Introduction

Quantum Electrodynamics (QED) states that the photon's rest mass should be zero, although an extremely small mass could exist that current experiments are unable to detect. Using the uncertainty principle, one can estimate a photon mass of the order of 10^{-66} g, taking the age of the universe as a reference. Although such a tiny mass would be practically impossible to measure, a massive version of QED is theoretically simpler [1] and provides a consistent framework to study the possible physical implications of a massive photon, such as variations in the speed of light [2], deviations from Coulomb's law [4] and Ampère's law [5], the existence of longitudinal electromagnetic waves [6], and an additional Yukawa-type potential for magnetic dipole fields [7].

Massive Proca electrodynamics is the simplest model in which the photon acquires a small mass, obtained by adding a mass term to the electromagnetic Lagrangian. The Proca field is described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here, M is interpreted as the photon mass and the scale M^{-1} corresponds to its reduced Compton wavelength, although the mass term breaks gauge invariance.

Cornwall [8] showed that, in the Jackiw–Johnson model [9], a symmetry-breaking mass term cannot be added without losing renormalizability, since it violates the Ward–Takahashi identities. Gauge invariance can be restored by introducing a nonlocal and non-polynomial term that preserves the symmetry. Following Cornwall's procedure, this chapter restores gauge invariance in Proca theory, analyzes its canonical structure, and determines the Hamiltonian with full gauge freedom, ultimately imposing suitable gauge conditions and computing the corresponding Dirac brackets.

Proca Field Model with Guage Symmetry

The Proca field described by the Lagrangian density (1) is not gauge invariant; however, gauge symmetry can be restored by adding a nonlocal and non-polynomial term to (1). If the mass term transforms as

$$\frac{1}{2}M^2 A_\mu A^\mu \to \frac{1}{2}M^2\left(A_\mu + \partial_\mu\Lambda\right)\left(A^\mu + \partial^\mu\Lambda\right), \tag{2}$$

and the parameter Λ is redefined as

$$\Lambda \to \theta \equiv -\frac{1}{e}\frac{1}{\Box}\partial_\mu A^\mu, \tag{3}$$

one to rewrite the mass term as

$$\frac{1}{2}M^2\left[A_\mu + \frac{1}{e}\partial_\mu\theta\right]^2. \tag{4}$$

This expression becomes gauge invariant under

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x), \qquad \theta(x) \to \theta(x) - e\Lambda(x), \tag{5}$$

with $\Box\theta \neq 0$, where θ is an auxiliary scalar field and e a coupling constant. This leads to the effective gauge-invariant Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M^2\left[A_\mu + \frac{1}{e}\partial_\mu\theta\right]^2. \tag{6}$$

From (6), the corresponding Euler–Lagrange equations follow:

$$\partial_\nu F^{\nu\mu} + M^2 A^\mu = -\frac{1}{e}M^2\partial^\mu\theta, \qquad \partial_\nu\left[A^\nu + \frac{1}{e}\partial^\nu\theta\right] = 0. \tag{7}$$

From the definition of the canonical momenta associated to the filed A_μ we obtain

$$\phi^1 \equiv \pi^0 = 0\,. \tag{8}$$

Let us define $p(x) \equiv \partial_0 S$, then in the context of the Hamilton-Jacobi formalism, we have the initial set of Hamilton-Jacobi Partial Differential Equations (HJPDE)

$$\begin{aligned} \phi^0 &\equiv p^t + \mathcal{H}_c = 0 \qquad &&, \to t, \\ \phi^1 &\equiv \pi^0 = 0 \qquad &&, \to \omega_1, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \mathcal{H}_c =& \frac{1}{2}\pi^k\pi^k + \frac{1}{2}\frac{e^2}{m^2}p_\theta^2 - \left(\partial_k\pi^k\right)A_0 - eA_0p_\theta + \frac{1}{4}F_{ki}F^{ki} \\ &+ \frac{m^2}{2}\left(A_k + \frac{1}{e}\partial_k\theta\right)^2. \end{aligned} \tag{10}$$

The fundamental Poisson brackets (PB) are defined by

$$\begin{aligned} \left\{A_\mu(x), \pi^\nu(y)\right\} &= \delta_\mu^\nu\delta^3(x-y), \\ \left\{\theta(x), p^\theta(y)\right\} &= \delta^3(x-y). \end{aligned} \tag{11}$$

Given the phase space of the theory, and by defining the PB we build the fundamental differential which characterizes the evolution of any function of the phase space as:

$$dF(x) = \int d^3y\left[\left\{F(x), \phi^0(y)\right\}dt + \left\{F(x), \phi^1(y)\right\}d\omega_1(y)\right]. \tag{12}$$

Now, it is necessary to verify the integrability conditions (IC), in order to ensure that the set HJPDE (9) is complete. The IC on the equations (9) imply,

$$d\phi^1 = \left[\partial_k^x\pi^k + ep_\theta\right]dt = 0. \tag{13}$$

which give new HJPDE:

$$\phi^2 \equiv \partial_k^x\pi^k + ep_\theta = 0. \tag{14}$$

Now, the the study of the IC of $\bar{\phi}^2$ will not give any new Hamiltonian density. Therefore, we end up with the following complete set of HJPDE,

$$\begin{aligned} \phi^0 &\equiv p^t + \mathcal{H}_c = 0, \\ \phi^1 &\equiv \pi^0 = 0, \\ \phi^2 &\equiv \partial_k\pi^k + ep_\theta = 0. \end{aligned} \tag{15}$$

Since ϕ^2 result from the IC of ϕ^1, it is not canonical and does not have a corresponding variable in the original phase-space. In this case we may expand de parameter space with a new parameter ω_2. Then the fundamental differential, which involves all the HJPDE has the following form,

$$dF(x) = \int d^3y\left[\left\{F(x), \phi^0(y)\right\}dt + \left\{F(x), \phi^1(y)\right\}d\omega_1(y) + \left\{F(x), \phi^2(y)\right\}d\omega_2(y)\right]. \tag{16}$$

From this equation we obtain the following set of characteristics equations

$$\begin{aligned} dA_\mu &= \delta_\mu^k\left(\pi^k - \partial_k A_0\right)dt + \delta_\mu^k\partial_k d\omega_2, \\ d\pi^\mu &= \left\{\delta_0^\mu\left(\partial_k\pi^k + ep_\theta\right) + \delta_k^\mu\left[\partial_l F^{lk} - m^2\left(A_k + \frac{1}{e}\partial_k\theta\right)\right]\right\}dt, \\ d\theta &= \left[\frac{e^2}{m^2}p_\theta - eA_0\right]dt + ed\omega_2\,, \\ dp_\theta &= \frac{m^2}{e}\left[\partial_k\left(A_k + \frac{1}{e}\partial_k\theta\right)\right]dt. \end{aligned} \tag{17}$$

This equations describe the dynamical evolution of the system depending on the parameters dt, τ_{01} and τ_{03}. Frobenius' theorem implies that these parameters are independent, therefore the evolution in the direction of a given parameter is independent of the evolution along the others. In the context of the theory, we propose to impose the following gauge-fixing conditions,

$$\phi^3 \equiv A_0 = 0 \qquad , \qquad \phi^4 \equiv \partial_k A_k + \frac{m^2}{e}\theta = 0. \tag{18}$$

These equations must be incorporated into the set of HJPDES defined by the fundamental differential (16), with each of them assigned an independent parameter. Then we define the dynamics of the system by the new fundamental differential,

$$\begin{aligned} dF(x) =& \int d^3y\left[\left\{F(x), \phi^0(y)\right\}dt + \left\{F(x), \phi^1(y)\right\}d\omega_1(y) + \left\{F(x), \phi^2(y)\right\}d\omega_2(y)\right] \\ &+ \int d^3y\left[\left\{F(x), \phi^3(y)\right\}d\omega_3(y) + \left\{F(x), \phi^4(y)\right\}d\omega_4(y)\right]. \end{aligned} \tag{19}$$

Now, the set of HJPDE $\left(\phi^1, \phi^2, \phi^3, \phi^4\right)$ is not in involution, since it is possible to construct the following matrix $M^{ij}(\mathbf{x}, \mathbf{y}) \equiv \left\{\phi^i(x), \phi^j(y)\right\}$, which turns out to be non-singular, with its inverse given by

$$M^{-1}(x,y) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{D_x} \\ -1 & 0 & 0 & 0 \\ 0 & \frac{1}{D_x} & 0 & 0 \end{pmatrix}\delta^3(x-y), \tag{20}$$

where $D_x \equiv \nabla_x - m^2$. The regularity of the matrix implies that the independent variables $(\omega_1, \omega_2, \omega_3, \omega_4)$ can be expressed as functions of the parameter t, namely

$$d\omega_i(y) = -\iint d^2u\,d^2v\,\Phi_{ij}^{-1}(y,v)\left\{\phi^j(v), \phi^0(u)\right\}dt. \tag{21}$$

Accordingly, the Generalized Brackets (GB) of the system can be defined as

$$\{F(x), G(y)\}^* \equiv \{F(x), G(y)\} - \int\int d^3u\,d^3v\,\{F(x), \phi^n(u)\}\left(M^{-1}\right)^{nl}(u,v)\left\{\phi^l(v), G(y)\right\}. \tag{22}$$

The only non-vanishing fundamental GB are given by

$$\left\{A_i(x), \pi^k(y)\right\}^* = \left(\delta_i^k + \frac{\partial_i^x\partial_k^x}{D_x}\right)\delta^3(x-y), \tag{23}$$

$$\{A_i(x), p_\theta(y)\}^* = -\frac{m^2}{e}\frac{\partial_i^x}{D_x}\delta^3(x-y), \tag{24}$$

$$\left\{\pi^i(x), \theta(y)\right\}^* = e\frac{\partial_i^x}{D_x}\delta^3(x-y), \tag{25}$$

$$\{\theta(x), p_\theta(y)\}^* = \left(1 + \frac{m^2}{D_x}\right)\delta^3(x-y)\,. \tag{26}$$

Conclusions

In this paper we have studies Proca Field with Guage Symmetry with the Hamilton-Hacobi method. We have shown that the set of HJPDE ϕ^1 and ϕ^2 are involutive and a gauge symmetry is associated with them. In order to eliminate the gauge freedom of the theory, the gauge conditions $A_0 = 0$ and $\partial_k A_k + \frac{m^2}{e}\theta = 0$ were imposed and the generalized brackets were calculated.

References

[1] A. Ignatiev and G. Joshi, Phys. Rev. **D53**, 984 (1996).

[2] A. Goldhaber and M. Nieto, Rev. Mod. Phys. **43**, 277 (1971).

[3] Liang-Cheng Tu, J. Luo, and G. Gilles, Rep. Prog. Phys. **68**, 77 (2005).

[4] E. R.Williams, J. E. Faller, and H. Hill, Phys. Rev. Lett. **26**, 721 (1971).

[5] M. A. Chernikov, C. J. Gerber, H. R. Ott, and H. J. Gerber, Phys. Rev. Lett. **68**, 3383 (1992).

[6] L. Bass and E. Schro¨dinger, Proc. R. Soc. London, Ser. **A 232**, 1 (1955).

[7] J. D. Barrow and R. R. Burman, Nature (London) **307**, 14 (1984). // E. Fischbach et al., Phys. Rev. Lett. **73**, 514 (1994).

[8] J. M. Cornwall, Phys. Rev. **D10**, 500, (1974).

[9] R. Jackiw and K. Jonhson, Phys. Rev. **D8**, 2386, (1973).