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A geometric–computational framework for teaching and visualising parallel transport in curved spaces

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ABSTRACT

Parallel transport is a fundamental concept in differential geometry and underpins the mathematical structure of spacetime in general relativity. However, its abstract formalism presents substantial cognitive challenges for students encountering relativity for the first time. This work introduces a geometric–computational framework for teaching parallel transport, complemented by a practical approach that enables systematic analysis and interactive visualisation in an educational setting. The proposed method generalises the intuitive notion of transport along geodesics to arbitrary curves by discretizing them into successive geodesic segments. Numerical simulations on spherical manifolds demonstrate that the approximation converges rapidly to the exact solution, with angular deviations becoming negligible even for moderate discretizations. By translating abstract geometric structures into interactive visualisations and computational exercises, the framework strengthens students' intuition, integrates digital tools into the learning of geometry and physics, and broadens access to advanced concepts – fostering a deeper engagement with the geometric essence of General Relativity. The article concludes with a detailed discussion of potential implications for physics and mathematics teaching, curriculum design, and future research, highlighting the method's broader relevance for both physics education and the practice of numerical modelling in curved spaces.

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KEYWORDS
Parallel transport; physics education; computational geometry; general relativity; numerical simulation

1. Introduction

General relativity remains one of the most profound achievements of modern physics, reshaping the understanding of space, time, and gravitation (Contreras et al., 2025; Guidry & Guidry, 2019; Khan, 2020; Renn, 2004). Beyond its foundational role in explaining astrophysical phenomena such as black holes (Dias et al., 2024; Franchini & Völkel, 2024; Santos, 2025), gravitational waves (Endlich et al., 2017; Krishnendu & Ohme, 2021; Weber, 2004), and the dynamics of the universe (Suntola, 2025), general relativity has also become increasingly relevant in everyday technologies, from GPS navigation systems to satellite communications (Ashby, 2003; Dimitrov, 2023). This dual character – as both a cornerstone of theoretical physics and a driver of practical innovation – highlights the importance

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of fostering a deeper and more accessible understanding of relativity in contemporary education (Gasparini et al., 2025; Herrmann & Pohlig, 2023; Jarosievitz & Sükösd, 2021; Kersting, 2022; Leonardi et al., 2022; Levrini, 2013; Postiglione & De Angelis, 2022; Prado et al., 2020).

Despite its remarkable predictive success, general relativity faces conceptual and empirical challenges that constrain its scope. Its incompatibility with quantum mechanics prevents the formulation of a fully consistent theory of quantum gravity – a necessary foundation for any prospective theory of everything capable of unifying the four fundamental interactions – while the existence of singularities, such as those associated with black holes and the origin of the universe, exposes intrinsic limits to its validity. Furthermore, the need to postulate dark matter and dark energy to account for cosmological observations suggests a possible incompleteness of the current gravitational framework. These difficulties are compounded by unresolved mathematical problems, including the global characterisation of the solution space of Einstein's equations and the rigorous definition of gravitational energy, as well as by the scarcity of experimental evidence in extreme regimes where significant deviations from the theory might emerge.

From an educational perspective, these challenges heighten the importance of instructional strategies that help students make sense of both the strengths and limitations of the theory. In recent years, physics education research has emphasised the value of approaching advanced concepts not solely as abstract theories, but as phenomena that learners can actively explore through digital tools, simulations, and interactive models (Султаналиева et al., 2021; Kabigting, 2021; Martinez et al., 2010; Morais et al., 2021; Pokhrel, 2024; Rehman et al., 2021; Turrubiartes et al., 2020; Yurchenko et al., 2023). For students in science and engineering, these approaches are essential to bridge mathematical formalism with physical intuition, while simultaneously fostering transversal skills such as critical thinking and problem-solving. Nevertheless, teaching general relativity remains pedagogically challenging: the abstract nature of curved spacetime and the subtleties of fundamental concepts – such as geodesics and parallel transport – make it difficult for many students to progress beyond procedural understanding when instruction relies exclusively on lectures or textbooks.

Extending this perspective, research on the teaching and learning of relativity at both the secondary and undergraduate levels has provided valuable insights into how students engage with these complex ideas (Alstein et al., 2021; Arriassecq & Greca, 2007; Berenguer & Selles, 2001; Christensen & Moore, 2012; Cramer, 2022; Jho, 2014; Pérez & Solbes, 2003, 2006; Postiglione & De Angelis, 2022; Velentzas & Halkia, 2013; Zapata et al., 2024a, 2024b). Studies have identified recurring difficulties, such as visualising spacetime curvature and interpreting relativistic effects as physically meaningful phenomena rather than purely mathematical constructs (Alias & Ibrahim, 2013; Jho, 2014; Kizilcik & Yavaş, 2017; Tanel, 2014). In response, a variety of instructional strategies have been explored, including interactive visualisations, virtual laboratories, and inquiry-based learning activities (Belloni et al., 2004; Carr et al., 2007; De Hosson et al., 2012; McGrath et al., 2008; Tarng et al., 2022). More recently, computational simulations have gained prominence as tools for active engagement, enabling learners to experiment with relativistic scenarios and develop a more tangible grasp of abstract principles (Alstein et al., 2023). While these efforts represent significant progress, there remains considerable potential to expand the range of resources for teaching advanced geometric notions – such as curvature, geodesics, and parallel transport – through approaches that combine mathematical rigour with accessibility and intuitive visualisation.

Among these notions, parallel transport stands out as both illustrative and conceptually challenging. It provides a rigorous description of how vectors evolve along curved surfaces or spacetimes, thereby capturing the geometric essence of motion in curved spaces and the principles of general relativity (Wald, 2010). Beyond its theoretical importance, it also exemplifies the pedagogical difficulty of bridging abstract differential geometry with intuitive understanding. For many learners, the main obstacle lies in visualising how changes in direction emerge solely from curvature, independent of external forces. When effectively taught, parallel transport can serve not only as a mathematical tool but also as a pedagogical gateway to geodesics, curvature, and broader applications across physics. Embedding this concept into educational practice, supported by computational visualisation, therefore represents both a challenge and an opportunity.

From this educational standpoint, a further advantage for contemporary teaching proposals lies in the integration of recent advances in computational geometry and numerical analysis. Novel discrete approaches to parallel transport have provided both theoretical insights and practical tools, with discrete approximations of geometric structures on manifolds attracting considerable attention. Such developments offer concrete ways to render abstract constructs computationally tractable and pedagogically accessible. Finite element methods, for instance, have been successfully employed to approximate scalar curvature in arbitrary dimensions, with convergence guarantees that underscore the reliability of these discretizations (Gawlik & Neunteufel, 2025). This body of work highlights a broader principle: continuous geometric processes can be effectively approximated by discrete schemes while preserving essential features within well-controlled error bounds.

In parallel, efficient numerical techniques for parallel transport itself have been introduced, with applications in both computational geometry and physics (Guigui & Pennec, 2022). These approaches emphasise the central role of the Levi-Civita connection in numerical settings by achieving high levels of accuracy. For instance, research such as (Berchenko-Kogan & Gawlik, 2024) has proposed a finite element approximation of the Levi-Civita connection and its curvature in two dimensions, demonstrating that classical concepts of differential geometry can be effectively translated into computational frameworks. Similarly, in applied domains such as shape analysis, parallel transport schemes have been adapted to handle manifold-valued data with efficiency and precision (Louis et al., 2017).

Within this framework, educational initiatives have increasingly explored how these computational advances can be transformed into interactive teaching strategies. Computational environments that integrate visualisation and numerical experimentation provide students with direct access to the underlying geometric ideas, bridging the gap between formal theory and applied computation. For instance, Mathematica-based implementations have been employed to illustrate the Levi-Civita connection by solving differential equations and displaying the results in an accessible manner (Yukita, 2017). This strategy emphasises direct visualisation of existing theory, allowing learners to connect abstract formulations with computationally generated outcomes. Beyond educational settings, discrete parallel transport has also been applied to enhance manifold learning algorithms, demonstrating how such methods can bridge theoretical geometry with practical data-driven applications (Budninskiy et al., 2019).

Although advances in discrete differential geometry and computational models of parallel transport have demonstrated notable methodological versatility and pedagogical potential, they often remain either overly abstract or computationally demanding for effective integration into introductory courses in physics and geometry. To address this challenge, this work proposes a method specifically developed for teaching parallel transport in curved spaces, which privileges conceptual clarity over algorithmic complexity. The core idea is that parallel transport becomes significantly more intuitive when restricted to geodesics – on the sphere, for example, represented by great circles – since it relies only on elementary Euclidean geometry and avoids the need for explicit differential calculations. Building on this perspective, an alternative scheme is introduced that approximates parallel transport through a stepwise decomposition into geodesic segments, thereby bridging intuitive geometric reasoning with more formal mathematical structures. In addition, the method is implemented in Python, allowing both a systematic evaluation of its validity and the generation of interactive visualisations that strengthen its pedagogical impact. Within the broader landscape of computational geometry, the proposed framework emphasises both accessibility and didactic value, offering learners a simplified yet rigorous entry point into the study of curved spaces.

The novelty of the contribution lies in its pedagogical orientation. First, it offers a constructive and approachable pathway for students to explore the fundamental ideas of general relativity through geometric visualisation rather than exclusively through differential equations. Second, it exemplifies a model of technology-enhanced physics education that aligns with contemporary calls for early exposure to frontier knowledge supported by active learning methodologies. Importantly, the proposed method is not limited to advanced university courses; it is designed for students beginning their undergraduate studies, for learners in secondary education, and even for those outside professional training in physics or engineering who wish to grasp the basic principles of relativity. In this sense, the method serves not only as a didactic tool but also as a vehicle for scientific dissemination, making an often academically restricted concept accessible to a broader public.

From a pedagogical standpoint, the proposed method also aligns with Ausubel's model of meaningful learning (Ausubel, 1983), as it is grounded in mathematical ideas that students have typically mastered long before encountering differential geometry. Crucially, parallel transport in flat spaces is an operation students perform implicitly whenever they compare vectors at different points – for instance, when computing derivatives of vector fields, which necessarily require bringing vectors to a common location. Making this implicit practice explicit helps learners recognise that they have been applying parallel transport all along, even if it was unnamed. Once this familiarity is established, extending the concept to transport along geodesics becomes a natural and accessible generalisation, providing a conceptual bridge between elementary vector reasoning and the geometric foundations of general relativity.

Building on this foundation, the method approximates transport along arbitrary curves by discretizing the path into a sequence of points and connecting them through geodesics (great circles), with convergence to the exact result as the number of subdivisions increases. This construction requires only elementary geometric notions – such as understanding the angle between vectors – making it accessible even to learners without an advanced mathematical background. The strategy yields a twofold benefit: (i) an algorithmic framework that can be easily implemented in open-source environments such as Python, avoiding reliance on specialised software, and (ii) a pedagogical tool that renders parallel transport a tangible, incremental process. In contrast to previous interactive visualisers that depend on solving and displaying transport equations (Postiglione & De Angelis, 2022), the present approach grounds the learning experience in a discrete geometric construction that is both computationally straightforward and visually intuitive.

This article is structured as follows. Section 2 establishes the necessary mathematical background, reviewing the concept of parallel transport and its fundamental distinction from Euclidean transport. Section 3 presents the formal framework for computing parallel transport on curved manifolds. Building upon this foundation, Section 4 introduces the proposed approximation scheme based on successive geodesic segments. Section 5 describes the implementation of numerical simulations and provides a comparative error analysis between the proposed and traditional approaches. Finally, Section 6 discusses the pedagogical implications and potential applications of this work, concluding with final remarks and directions for future research.

2. Methods

This study adopts a design-based research approach to develop and evaluate a novel pedagogical framework for teaching parallel transport on curved surfaces. The methodology focuses on developing a geometrically grounded and conceptually clear approach that extends the intuitive idea of transport along geodesics to arbitrary curves. Computational simulations were used to illustrate and validate the method, serving as tools to support visualisation and exploration. The primary contribution, however, lies in the design of an approach that bridges abstract geometric formalism with accessible educational representations.

The methodology is structured into four components: (A) theoretical framework, (B) computational implementation, (C) validation of the numerical scheme, and (D) pedagogical considerations. Each component is designed to support the demonstration, assessment, and potential instructional use of the proposed approach, thereby bridging abstract geometric formalism with conceptual and visual understanding.

2.1. Theoretical framework

The theoretical foundation involves a review of differential geometry, with particular emphasis on the properties of parallel transport on curved surfaces such as the two-dimensional sphere. This component establishes the mathematical and conceptual foundations for the proposed method, ensuring it maintains formal rigour while remaining conceptually accessible. The focus was on defining the problem and outlining a general strategy for extending transport along geodesics to arbitrary curves, thereby providing a solid basis for subsequent computational implementation and visualisation.

2.2. Computational implementation

The computational implementation translates the theoretical method into a practical framework for exploration and illustration. Python was used for this purpose, leveraging libraries for symbolic and numerical computation. This computational component enabled the approach to be tested across various trajectories and conditions, facilitating the assessment of its conceptual soundness and numerical reliability. Importantly, the implementation serves as a tool to visualise and explore the geometric behaviour of parallel transport, rather than constituting the main objective of the study.

2.3. Validation of the numerical scheme

Validation was performed using test cases with known or analytically solvable results to evaluate the consistency and convergence of the approach. Deviations from exact solutions were analyzed to confirm the method's reliability and robustness. This component provided quantitative support for the method, demonstrating that the geometrically grounded strategy produced results consistent with theoretical expectations.

2.4. Pedagogical considerations

A pedagogical framework was developed to integrate the proposed method and its simulations into diverse educational settings, including secondary, undergraduate, and informal learning environments. This framework emphasises interactive visualisation and inquiry-based exploration to make abstract concepts – such as curvature and parallel transport – accessible and engaging. A collection of open-access resources, including Python code, Jupyter notebooks, and classroom activities, was created to support implementation and foster collaboration among educators and researchers.

3. Parallel transport and its role in the geometry of spacetime

Parallel transport is a cornerstone of modern physics, providing a direct link between the geometry of curved spaces and observable physical phenomena. In general relativity, it describes the evolution of vectors – such as velocity or momentum – along trajectories in curved spacetime, forming the basis for understanding effects like orbital precession and the deflection of light by massive bodies. Furthermore, it offers a precise definition of geodesics, characterising them as paths along which a vector remains constant under transport. This foundational concept underpins more advanced geometric constructs, including curvature, frame dragging, and the Riemann tensor. Beyond relativity, parallel transport finds critical applications in diverse fields such as gauge field theory, materials science, and computational geometry, where it enables the analysis of vector and tensor behaviour on curved manifolds. In essence, parallel transport quantifies the evolution of directional information in curved spaces, thereby bridging abstract geometry with tangible physics.

A paradigmatic example is the motion of a free particle in a gravitational field. Its trajectory follows a geodesic – the analog of a straight line in curved spacetime – along which its velocity vector is parallel transported. This process ensures a locally consistent definition of the particle's direction of motion. Other physical attributes, such as spin orientation or the axis of a gyroscope, evolve under related principles like Fermi-Walker transport, giving rise to measurable effects such as geodetic precession. These cases illustrate that parallel transport is not merely a mathematical abstraction, but a fundamental mechanism through which the geometry of spacetime manifests in physical observation.

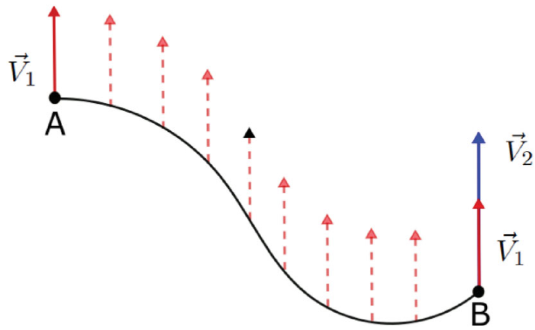

Figure 1. Parallel transport in flat Euclidean space. A vector at point $A(\overrightarrow{V_1})$ is moved along the curve, remaining parallel to itself at each step (red dashed arrows). Upon reaching point B, its orientation is unchanged ($\overrightarrow{V_2} = \overrightarrow{V_1}$), demonstrating that in flat space, closed paths do not produce a net rotation, consistent with zero curvature.

Insight can be gained by first considering flat geometry. In Euclidean space, tangent spaces at different points can be naturally identified, and vectors may be displaced along a curve without altering their length or orientation. Graphically, this process can be imagined as sliding a vector along the curve (red dashed arrows in Figure 1), always parallel to itself. For a closed path, the vector returns to its original orientation, confirming the absence of curvature.

On curved manifolds, however, this global identification fails. Even infinitesimally close points possess tangent planes that are not parallel in the Euclidean sense. Transporting a vector around a closed loop generally produces a mismatch between the initial and final orientations. This deviation encodes the intrinsic curvature of the manifold and provides a direct geometric manifestation of gravitation itself.

4. Theoretical foundations and computational approach to parallel transport

As discussed in the previous sections, parallel transport provides a formal procedure for moving a vector along a curve on a curved surface – or more generally, within curved spacetime – while preserving its direction according to the local geometry. In flat spaces, this corresponds to keeping the vector strictly parallel, but in curved spaces the operation represents a generalisation of the notion of ‘parallelism’, defined relative to the underlying curvature.

In the context of general relativity, this operation is described using the covariant derivative, which specifies how vectors change as they move through curved space. The Levi-Civita connection, a metric-compatible and torsion-free connection, provides the specific mathematical framework for implementing this transport. It ensures that the process is consistent with the geometry of the manifold and the physical principles underlying spacetime curvature.

Formally, if a vector V^μ is transported along a parametrised curve $x^\mu(\lambda)$, parallel transport requires that its covariant derivative vanish along the curve:

$$\frac{DV^\mu}{D\lambda} = 0 \tag{1}$$

Here, V^μ denotes the transported vector, with the index μ labelling its components in a chosen coordinate system. The parameter λ represents the curve's evolution variable (e.g. proper time, arc length, or another convenient parameter). The operator $\frac{D}{D\lambda}$ denotes the covariant derivative along the curve, defined in coordinates as:

$$\frac{DV^\mu}{D\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\upsilon\rho} \frac{dx^\nu}{d\lambda} V^\rho, \tag{2}$$

which extends the ordinary derivative by incorporating curvature effects, thereby distinguishing genuine changes in the vector from those that are mere artifacts of the coordinate system.

Intuitively, this condition ensures that a transported vector maintains its orientation relative to the local geometry. In flat space, tangent spaces can be globally identified, and vectors can be compared directly. By contrast, in curved manifolds direct comparisons are not possible; they are only meaningful when mediated by the connection, which guarantees that parallelism is preserved locally even in the absence of a global standard.

In coordinate form, the parallel transport condition becomes a system of first-order coupled differential equations:

$$\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\upsilon\rho} \frac{dx^\nu}{d\lambda} V^\rho = 0, \tag{3}$$

where $\Gamma^\mu_{\upsilon\rho}$ are the Christoffel symbols, which act as the coefficients of the Levi-Civita connection in a metric space. This connection is uniquely determined by the metric $g_{\mu\nu}$, which defines the local geometric properties of the space or spacetime. The indices ν and ρ denote summation over components (Einstein summation convention), reflecting the way the manifold's curvature couples the evolution of different vector components. Importantly, geodesics – the 'straightest' possible paths in curved geometry – are defined by the condition that their tangent vectors are parallel transported along themselves.

For most realistic curved manifolds or spacetime geometries, closed-form analytical solutions to this system are not feasible. As a result, computational methods are indispensable. The typical approach is to discretize the curve into small steps of size $\Delta\lambda$ and incrementally update the vector according to the transport law. A simple yet often effective approximation is the Euler scheme:

$$V^\mu(\lambda + \Delta\lambda) \approx \; V^\mu(\lambda) - \Gamma^\mu_{\upsilon\rho}(x(\lambda)) \frac{dx^\nu}{d\lambda} V^\rho(\lambda)\Delta\lambda. \tag{4}$$

In practice, more accurate integration algorithms, such as Runge–Kutta methods, are typically employed to minimise error accumulation and preserve geometric consistency. Through this process, one obtains the trajectory of the transported vector along the curve, revealing how curvature induces rotations in its orientation.

This formalism underscores the mathematical and computational sophistication required to handle parallel transport rigorously. While indispensable for theoretical physics and advanced applications, it presents a significant barrier to its adoption in early educational contexts. The reliance on differential geometry, tensor calculus, and numerical integration exceeds the typical scope of secondary or early undergraduate curricula, thus framing the pedagogical challenge that motivates the alternative approach developed in the next section.

5. An alternative approximation scheme for parallel transport: discretization of a curve into geodesic segments

This section presents an alternative approximation scheme for teaching parallel transport, based on discretizing arbitrary curves into sequences of geodesic segments. This approach is designed to provide a geometrically intuitive and conceptually clear method for visualising vector transport on curved surfaces, while maintaining formal consistency with the underlying mathematical framework. The section begins by outlining the geometric method on the sphere, highlighting its conceptual foundations. Next, the discretization procedure is introduced, demonstrating how arbitrary curves can be approximated by concatenated geodesic segments. Finally, the pedagogical significance and potential applications of the proposed approach for instruction in advanced geometry and physics courses are discussed.

5.1. Geometric method on great circles of a sphere

To understand the parallel transport of a vector on a sphere, it is helpful to begin with the simplest case: motion along a great circle. A great circle – a geodesic formed by the intersection of the sphere with a plane passing through its centre – has the special property that the tangent planes at different points along the path all rotate about a single, fixed axis. This shared reference allows the vector to maintain a constant orientation relative to that axis, making the process accessible through basic geometric intuition and simple visual aids such as a protractor, as illustrated in Figure 2.

this example, the blue vectors represent the common reference axis, shown here as the vertical z-axis. Parallel transport is carried out by placing the vector in each tangent plane while preserving both its magnitude and the angle α it forms with the reference axis. As shown in Figure 2, the red vector is moved from point $\boldsymbol{P}$ to point $\boldsymbol{Q}$ along the great circle, maintaining the same angle $\boldsymbol{\alpha}$ throughout its motion.

When the trajectory of a moving object on the sphere does not follow a single great circle, the situation becomes more complex and cannot be fully captured through direct visualisation. Vectors such as velocity, which are defined within the tangent plane at each point along the curve, are in general not transported strictly along great circles. As shown in Figure 3, these tangent planes vary smoothly from point to point as the vector moves across the surface, making it impossible to define a single, globally consistent reference for parallelism.

This variability in tangent planes leads to qualitatively different outcomes depending on whether the trajectory is open or closed. Along an open path, the vector ends with an orientation determined entirely by the route taken. In contrast, when the path forms a closed loop, the vector returns to its starting point, allowing its initial and final orientations to be directly compared. On a curved surface such as a sphere, these two orientations typically differ, providing a visible and intuitive signature of curvature. This distinction between open and closed paths serves as a powerful way to connect local geometric changes along the path with the global properties of the surface.

The case of great circles on the sphere provides a geometric framework that enables direct visualisation and conceptual understanding of how parallel transport operates in a curved space. Along such paths, the evolution of a vector can be readily interpreted,

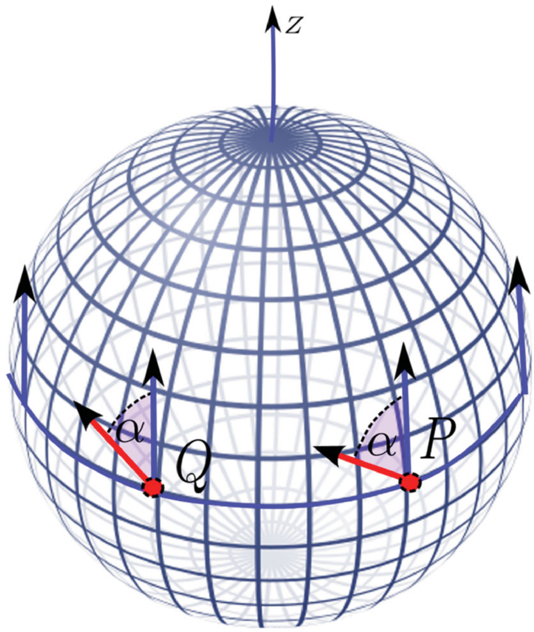

Figure 2. Parallel transport along a great circle on a sphere. The tangent planes at different points along the curve share a common reference axis (vertical *z*-axis), allowing the vector to maintain a constant orientation relative to this axis during transport.

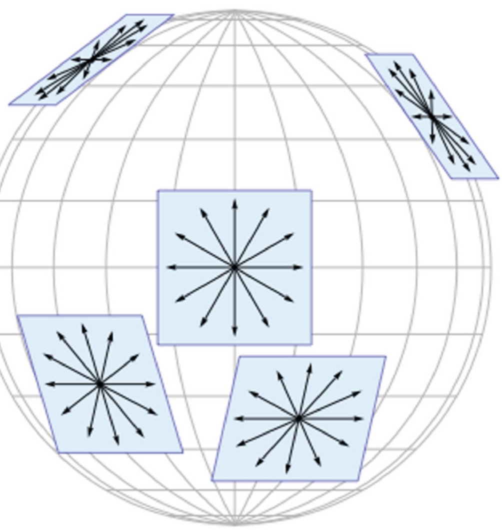

Figure 3. Illustration of tangent planes at different points on the sphere. Each plane touches the sphere at exactly one point, defining the local space in which vectors are represented – a concept that naturally extends to other curved surfaces. As the vector moves along the surface, these planes rotate relative to one another, highlighting the absence of a single, globally consistent reference for parallelism.

offering an accessible introduction to the underlying concept. However, when the trajectory follows a general curve, the process becomes significantly more complex and less straightforward to represent visually. To address this challenge, the next section introduces a discretization method that divides an arbitrary curve into small geodesic segments, providing a systematic and computationally efficient way to approximate and visualise parallel transport along complex paths on the sphere.

5.2. Approximation by discretizing a curve into geodesic segments

The previous section introduced parallel transport along great circles on the sphere, a situation where the geometry is straightforward to visualise and the procedure can be carried out entirely through geometric reasoning. This special case provides an accessible entry point for understanding how a vector's orientation changes when it moves across a curved surface. However, as mentioned before, in realistic scenarios, vectors often need to be transported along paths that are not geodesics. In these cases, the process becomes significantly more complex and typically requires advanced mathematical tools that are not introduced in basic undergraduate courses.

To address this challenge, a pedagogical strategy is proposed that extends the intuitive understanding developed for geodesics to more general curves. The central idea is to approximate an arbitrary curve by dividing it into a sequence of short geodesic segments that connect consecutive points along the path. Since the procedure for parallel transport along a single geodesic is already conceptually clear, students can use the same geometric reasoning to approach more complex trajectories without immediately relying on the formalism of covariant derivatives, Christoffel symbols, or coupled differential equations. This strategy bridges intuitive, visual reasoning with rigorous concepts, gradually preparing students for the formalism of differential geometry.

This approach is illustrated in Figure 4. Suppose a vector is to be transported parallelly from point A to point B along the curved path shown on the sphere. Directly performing this transport along the original curve would require advanced computations that are beyond the scope of introductory courses. Instead, the curve is approximated by dividing it into a small number of geodesic segments that serve as intermediate steps.

The initial tangent vector is shown in purple. The green vector represents the reference solution, obtained from the exact parallel transport along the original curve. The red vector corresponds to the approximate result obtained by transporting the vector step-by-step along the geodesic segments. In Figure 4, the curve has been divided into only two geodesic segments, leading to a significant discrepancy between the exact and approximate results, with an angular deviation of approximately $\Delta\phi \approx 64.42^\circ$.

As shown in Figure 5, increasing the number of geodesic segments greatly improves the approximation. In this case, the same curve is divided into seven shorter segments, reducing the angular deviation to $\Delta\phi \approx 5.88^\circ$. This progression illustrates how the approximation converges toward the exact solution as the number of segments increases. In the theoretical limit of infinitely many segments, the discretized geodesic construction becomes exact. This behaviour can be explored computationally, offering students a powerful and interactive way to experiment with the concept of convergence while deepening their understanding of parallel transport.

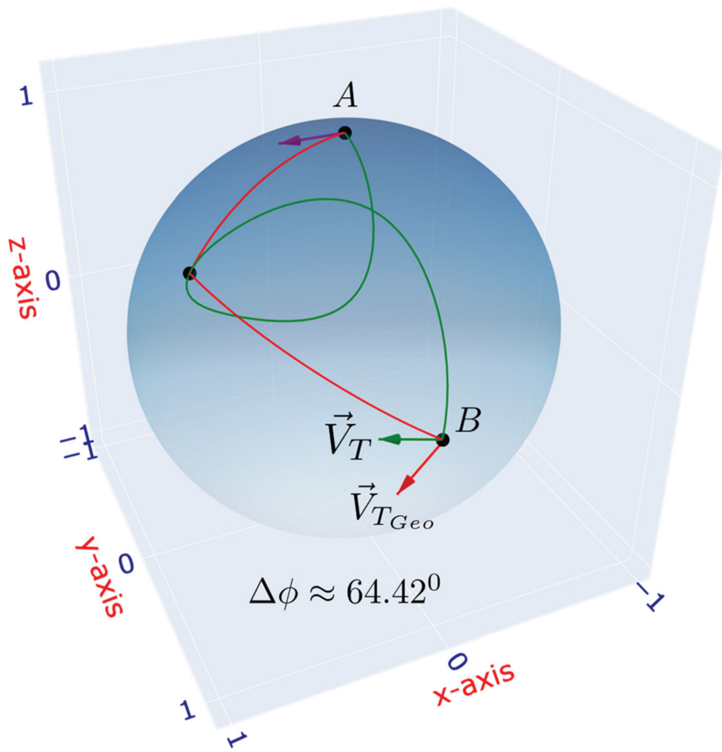

Figure 4. Parallel transport from point *A* to point *B* along a curved trajectory on the sphere. The path (green) is approximated using short geodesic segments (red), providing a simpler, stepwise method for visualising and computing the process. The final transported vector, ($\overrightarrow{V_T}$), differs from the geodesic transport vector, ($\vec{V}T_{Geo}$) by an angle $\Delta\phi \approx 64.42°$.

Beyond its computational value, this method highlights the geometric essence of parallel transport. By leveraging students' prior intuition about geodesics, the method offers a pathway to understanding how vectors evolve along curved trajectories on the sphere without requiring immediate engagement with abstract formalism. While the method is presented here in the context of the sphere, its structure provides a general strategy for approximating and visualising parallel transport on complex curves within this setting.

6. Numerical evaluation of the proposed approximation scheme

The geometric framework described previously was implemented computationally using Python with two main objectives: to validate the accuracy of the proposed approximation scheme for parallel transport, and to provide a visual and interactive environment that leverages modern technological resources to enhance understanding. This implementation enables users to visualise the entire procedure step by step, connecting abstract geometric ideas with concrete computational representations.

Beyond its role as a numerical testbed, the simulation functions as a pedagogical resource. It allows students to experiment with different curves, vary the number of geodesic subdivisions, and immediately observe the effects on the transported vector.

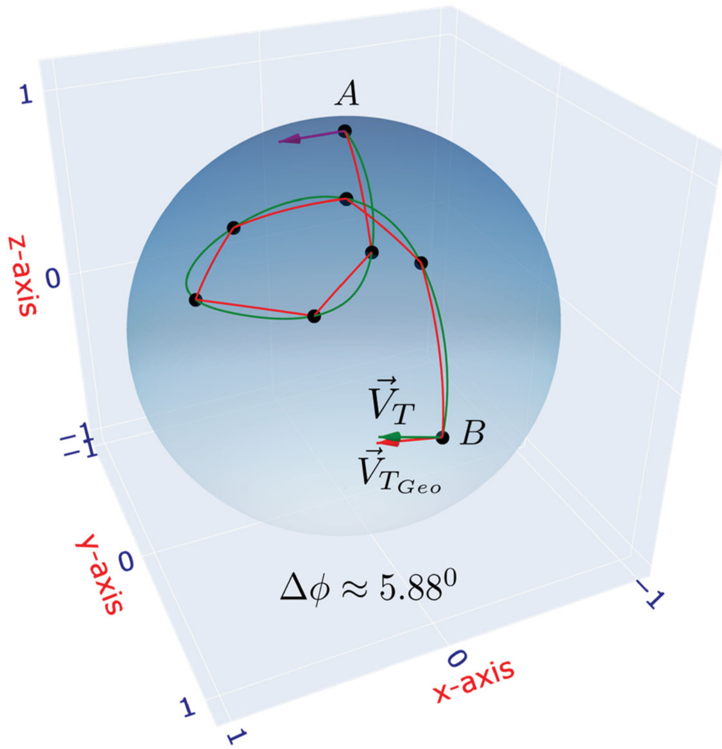

Figure 5. Parallel transport from point *A* to point *B* along the same curved trajectory on the sphere shown in Figure 4, but now using a finer approximation with additional geodesic segments (red). By increasing the number of intermediate points, the stepwise method becomes a closer representation of the continuous transport process. The discrepancy between the transported vector, ($\overrightarrow{V_T}$), and the geodesic transport vector, ($\vec{V}T_{Geo}$), is reduced to $\Delta\phi \approx 5.88^\circ$.

This dynamic and interactive approach supports deeper conceptual understanding while integrating seamlessly into contemporary teaching and learning contexts, where technology is crucial for making advanced mathematical concepts more accessible, intuitive, and engaging.

The process begins by defining a smooth reference curve on the sphere, which can be open, connecting two fixed points, or closed, forming a continuous loop. Once the curve is specified, it is discretized into N equally spaced points. The value of N is adjustable, allowing for systematic analysis of convergence.

Between each pair of consecutive points, a geodesic segment – corresponding to the great circle arc connecting them – is constructed. A vector placed at the initial point of the curve is then transported step by step along this network of geodesics. At each step, parallel transport is computed using the geometric method introduced earlier, where the transport rule is determined by the common axis of the tangent planes at neighbouring points along the geodesic.

To quantify the accuracy of the method, the geodesic-based approximation is compared with the exact parallel transport along the original curve. The exact solution is obtained by numerically solving the parallel transport differential equation (Equation

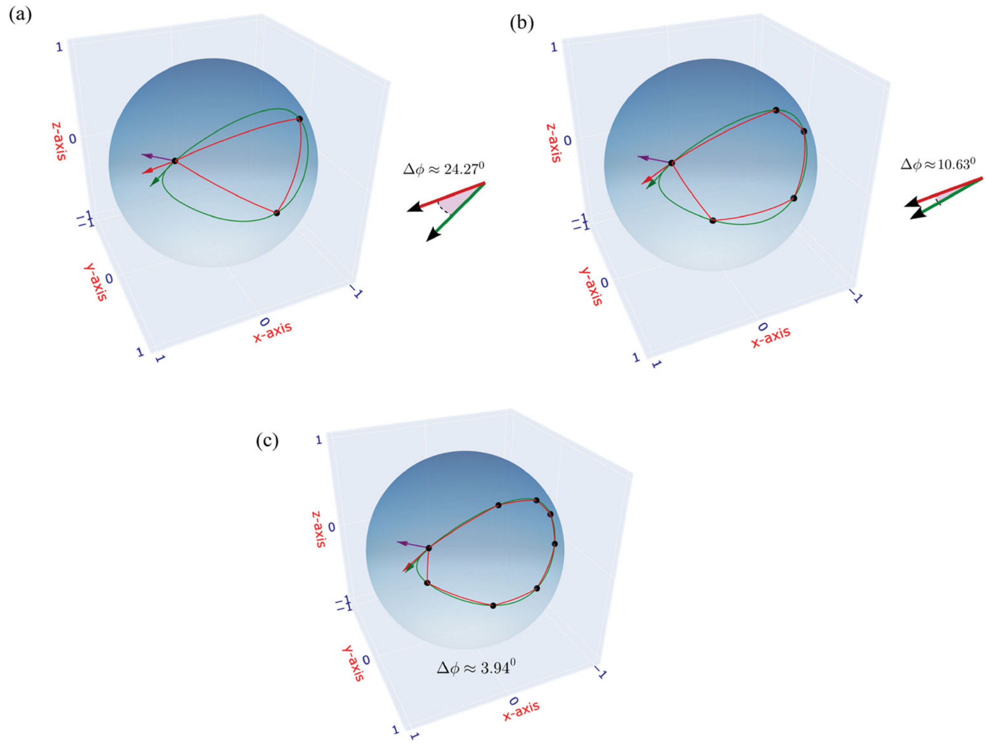

Figure 6. Parallel transport on the sphere approximated by discretized geodesic segments. The exact transport (green) and the stepwise approximation (red) are compared for increasing numbers of subdivisions: (a) $\Delta\phi \approx 24.27^\circ$. (b) $\Delta\phi \approx 10.63^\circ$., and (c) $\Delta\phi \approx 3.94^\circ$. These results, obtained from a Python numerical implementation, highlight how the discretized construction converges toward the exact solution, providing a direct comparison between the theoretical formulation and our proposed method.

3), which expresses the covariant derivative of the vector along the curve, using a standard Runge–Kutta integration scheme. As the number of geodesic subdivisions increases, the approximation converges toward the exact solution, consistent with the continuous formulation.

For non-geodesic curves, the method yields an approximation whose accuracy improves systematically with an increasing number of geodesic subdivisions. This convergence behaviour validates the discretization approach and confirms its consistency with the continuous formulation.

To illustrate the implementation and assess its accuracy, Figure 6 presents representative simulations of parallel transport on the sphere. Each panel shows a curve discretized into short geodesic segments (marked by black points). The initial tangent vector is depicted in purple. The reference solution, obtained from exact parallel transport along the curve, is shown in green, while the approximate result from the discretized geodesic construction is shown in red. The angular deviation $\Delta\phi$ between the red and green vectors quantifies the accuracy of the approximation, demonstrating improvement as the number of subdivisions increases.

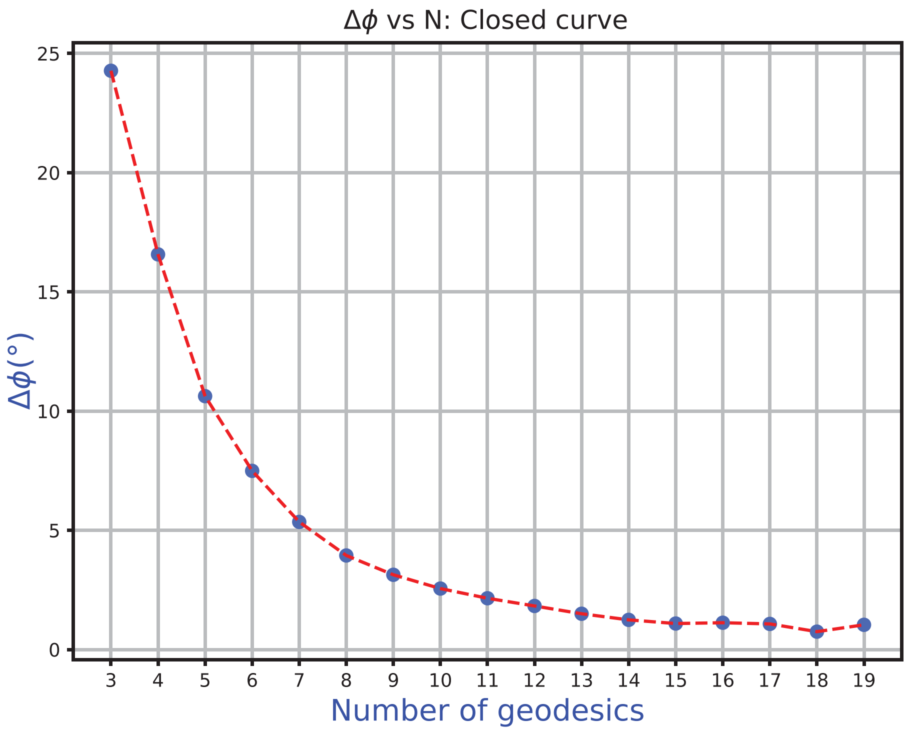

Figure 7. Angular deviation $\Delta\phi$ as a function of the number of geodesic segments N used to approximate a closed curve on the sphere. The numerical results (blue dots) obtained from the Python implementation show that the discrepancy decreases rapidly as N increases, approaching zero for finer discretizations. This behaviour confirms the convergence of the proposed stepwise geodesic method toward the exact parallel transport.

The panels in Figure 6 illustrate this convergence explicitly. With a coarse discretization, deviations remain noticeable: $\Delta\phi \approx 24.27^{\circ}$in panel (a) and $\Delta\phi \approx 10.63^{\circ}$in panel (b). By contrast, in panel (c), where the curve is subdivided more finely, the deviation decreases substantially to$\Delta\phi \approx 3.94^{\circ}$. In this case, the transported vectors are not explicitly displayed to avoid visual overlap. These examples highlight two key aspects of the method: (i) the algorithm preserves the geometric intuition of transporting vectors along successive geodesics, and (ii) the numerical discrepancy decreases systematically with finer discretization, illustrating convergence toward the exact transport.

To complement the visual examples in Figure 6, which illustrate the discretization process for 3, 5, and 8 geodesic segments, a systematic quantitative analysis was performed. At higher numbers of subdivisions, the discretized path becomes visually indistinguishable from the exact curve, providing no additional information in the three-dimensional representation. Therefore, Figure 7 presents the angular deviation $\Delta\phi$ as a function of the number of geodesic subdivisions. The results show that the deviation decreases rapidly as the discretization is refined, becoming negligible for approximately $N \gtrsim 12$, where the discrepancy approaches zero within numerical precision. This demonstrates that the method achieves reliable accuracy with a moderate computational cost.

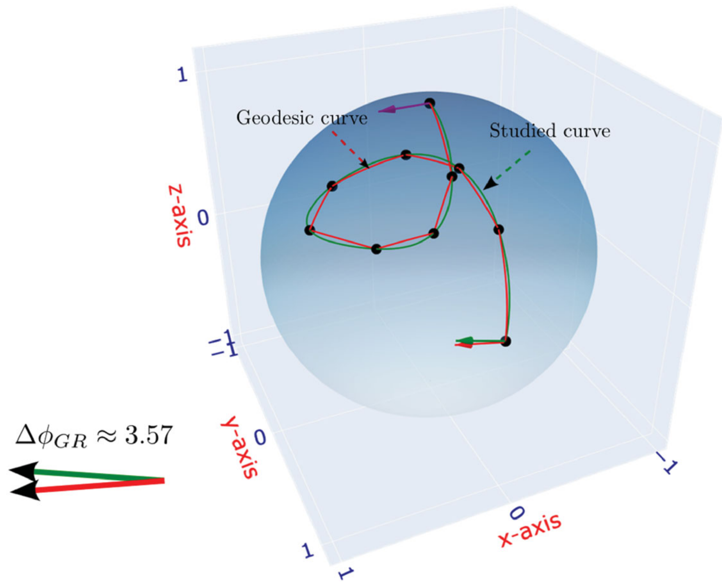

Figure 8. Parallel transport along a general (non-closed) trajectory on the sphere. The studied curve (green) is discretized into short geodesic segments (red), and the initial tangent vector (purple) is transported along it. The exact result of parallel transport is shown in green, while the geodesic-based approximation is shown in red. The angular deviation between both results is $\Delta\phi \approx 3.57°$,illustrating the accuracy of the numerical implementation.

In addition to closed curves, the scheme was also tested on open trajectories, as illustrated in Figure 8, the curve under study (green line) is discretized into short geodesic segments (red lines). The parallel transport of the initial tangent vector (purple) is then computed. The exact transport result is shown in green, while the red vector corresponds to the geodesic-based approximation. The angular difference between them, $\Delta\phi \approx 3.57°$, corresponds to a discretization into five geodesic segments, as shown in the figure, and provides a direct measure of accuracy. A systematic analysis for this case, shown in Figure 9, confirms the same convergence trend observed for closed curves: coarse discretizations yield noticeable deviations, but the error decreases rapidly as the number of segments increases.

Taken together, these results demonstrate that the proposed algorithm serves as both a consistent numerical method and an intuitive, visually clear framework for understanding parallel transport.

7. Pedagogical considerations

The computational framework and error analysis presented above demonstrate the robustness and accuracy of the approximation scheme for parallel transport. While these findings hold clear theoretical and numerical significance, their value extends far beyond computational physics. By transforming a highly abstract concept – traditionally accessible only through advanced mathematical formalism – into a dynamic, visual, and interactive experience, this methodology opens new possibilities for teaching and learning.

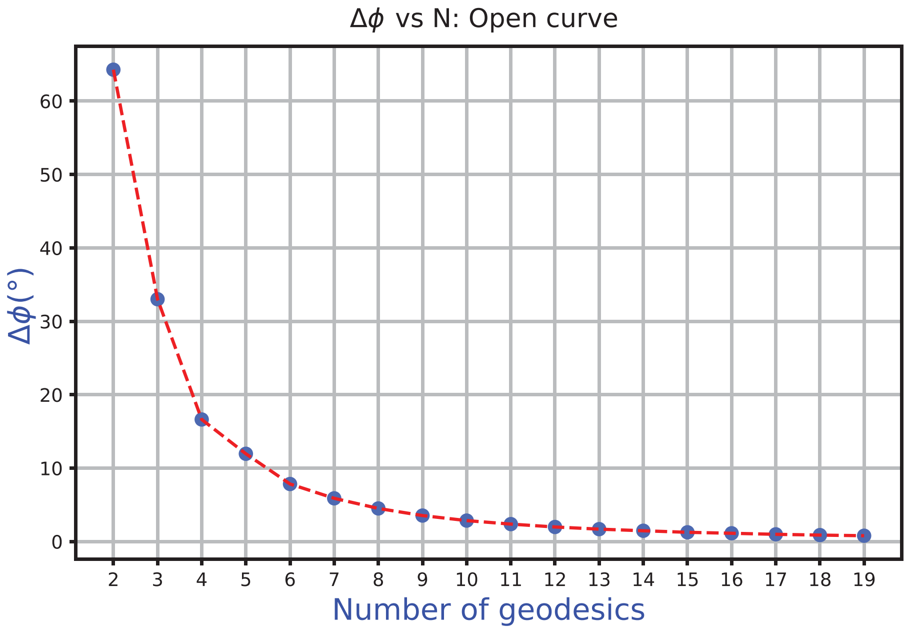

Figure 9. Angular deviation $\Delta\phi$ as a function of the number of geodesic segments N for an open curve on the sphere. The numerical results (blue dots) obtained from the Python implementation follow the expected convergence trend: coarse discretizations lead to large deviations, but the error decreases rapidly as N increases, approaching zero for finer approximations.

A central element of the educational design underpinning this proposal is its alignment with Ausubel's theory of meaningful learning (Ausubel, 1983; da Silva, 2020). According to this framework, learners develop deep understanding when new concepts can be anchored to pre-existing cognitive structures through appropriately designed conceptual organisers. In this regard, the notion of parallel transport connects naturally with intuitions that students have already internalised, often without explicit awareness: whenever they compare vectors – such as forces, velocities, or displacements – at different points in flat space, they implicitly perform a form of parallel transport. Making this implicit reasoning explicit provides a natural bridge to understanding parallel transport in curved spaces. The simulation developed in this work strengthens this bridge by offering an intuitive visual environment in which students can explore the qualitative behaviour of parallel transport before encountering its formal mathematical definition.

The pedagogical dimension of this work lies in its ability to bridge theory and hands-on exploration. Rather than functioning solely as a numerical tool, the simulation was purposefully designed as a flexible educational resource. It enables learners to directly engage with fundamental concepts such as curvature, geodesics, and vector evolution. Through interactive experimentation, students and educators can visualise these ideas in ways that are intuitive, conceptually rich, and cognitively meaningful.

To maximise its educational impact, the simulation is accompanied by a comprehensive set of open-access resources, as described in the Resource Availability section. These resources include the complete Python source code, interactive Jupyter notebooks, ready-to-use classroom activities, and detailed documentation tailored for educators and outreach facilitators. By combining pedagogical design with open accessibility, this project supports both localised implementation and global collaboration.

The following subsections illustrate how the methodology can be applied across three complementary educational contexts: (i) Applications in Secondary Education, where students are introduced to the core ideas of curvature and parallel transport with minimal mathematical prerequisites; (ii) Applications in Undergraduate Education, where formal mathematical tools are integrated with interactive conceptual exploration; and (iii) Informal and Outreach Education, where the focus shifts to public engagement and fostering scientific literacy through interactive demonstrations. Finally, Section (iv) Resource Availability and Open Access, providing an interactive web-based simulation of parallel transport that is freely accessible to educators, students, and researchers.

This integrated framework demonstrates how a tool originally developed for computational research can simultaneously advance the teaching and learning of advanced topics in modern physics, transforming abstract theory into meaningful, hands-on exploration.

7.1. Applications in secondary education

At the secondary level, introducing topics from modern physics often faces two obstacles: limited mathematical preparation and the perception that concepts such as general relativity are inaccessible until advanced study. The proposed approach addresses these barriers by providing a visual and exploratory gateway into the geometry of curved spaces. Students can manipulate vectors and trajectories directly, observing how curvature alone determines the evolution of direction, without needing to solve equations or work with formal differential geometry.

From the perspective of Ausubel's theory of meaningful learning, these interactive experiences serve as advance organisers (Ausubel, 1960), allowing students to anchor new concepts to their existing cognitive structures – such as prior knowledge of vectors, linear motion, and spatial reasoning – before formalising more abstract ideas. By engaging directly with the phenomena, learners construct preliminary mental models that facilitate the subsumption of subsequent, more complex principles of curved space.

This form of inquiry-based learning aligns with STEAM education principles by connecting mathematics, physics, and computational thinking in a single activity. Teachers can frame the simulation within thematic units, for example, exploring 'spacetime and motion' or 'hidden geometries in nature.' Activities could include prediction tasks ('What do you think will happen to the vector as it moves along this curved path?'), experimentation through interactive manipulation, and reflection prompts that encourage metacognition about how curvature differs from forces.

Importantly, each of these activities not only introduces foundational concepts of relativity but also actively engages students in meaningful learning processes, as they relate new visual and dynamic experiences to prior knowledge. Such implementations cultivate transversal skills such as hypothesis formulation, data interpretation, and scientific argumentation. Moreover, these activities can be integrated into extracurricular science clubs or enrichment modules, broadening access to students with varying levels of prior preparation while reinforcing the principles of cognitive anchoring central to Ausubel's framework.

It is worth emphasising that the activities proposed for this level do not require engagement with advanced mathematical tools. The simulation is specifically designed to replace formal derivations with intuitive visual reasoning, relying only on elementary geometric ideas that students already encounter in early mathematics courses – such as understanding vectors as entities with magnitude and direction and recognising angles relative to a reference frame. The notions of a tangent plane and a geodesic are introduced only at an intuitive level, without formal definitions. By grounding the learning process in these accessible concepts, the approach remains fully appropriate for secondary education and supports meaningful learning in the sense proposed by Ausubel, allowing students to integrate new qualitative experiences of curvature with their existing cognitive structures.

7.2. Applications in undergraduate education

At the undergraduate level, the simulation serves as a conceptual bridge between introductory physics courses and the formalism of general relativity. In many curricula, students encounter vector calculus and linear algebra long before tensor analysis, yet they rarely have opportunities to visualise how these mathematical tools describe curved manifolds.

The feasibility of implementing this activity at the undergraduate level rests on the fact that it draws directly on conceptual tools students typically acquire early in their training. The method relies only on basic ideas – such as interpreting vectors as quantities with magnitude and direction, identifying their orientation relative to a coordinate axis, and visualising tangent planes on the sphere – which are standard components of introductory physics and calculus courses. Moreover, comparing vectors at different points, a fundamental step in defining derivatives, implicitly involves a parallel-transport-like operation in flat spaces. By activating prior knowledge in accordance with Ausubel's theory of meaningful learning, the activity becomes conceptually accessible for undergraduate instruction.

The proposed method provides a progressive learning pathway designed to build upon students' prior knowledge and promote meaningful conceptual integration:

(1) Exploration phase – Students interact with the simulation to develop intuition about how vectors evolve along geodesics, connecting visual patterns to physical principles. This hands-on engagement acts as an advance organiser, enabling learners to anchor new geometric concepts to existing cognitive structures such as vector manipulation and spatial reasoning.
(2) Analytical phase – The same scenarios are revisited through simplified calculations, allowing learners to verify and generalise their observations. According to Ausubel, such progressive refinement of understanding strengthens cognitive anchoring, as learners relate experiential insights to formal mathematical representations.
(3) Formalisation phase – Once foundational intuition is established, students engage with tensor-based definitions of the Levi-Civita connection and parallel transport. The prior experiential grounding facilitates subsumption, allowing abstract formalism to integrate seamlessly into the learner's cognitive structure rather than being perceived as disconnected or purely symbolic.

The tool can be incorporated into general relativity courses, computational physics laboratories, or geometry-focused electives. Problem sets might require students to:

- Design their own geodesic paths and analyze the resulting vector behaviour.
- Compare the discrete approximation scheme with classical differential equation solutions.
- Extend the code to include alternative manifolds or curvature models.

Each of these activities leverages the principle of meaningful learning, as students actively relate procedural tasks to conceptual understanding, reinforcing both theoretical insight and computational skill.

By engaging with both computational modelling and visualisation, students develop integrated competencies in physics, programming, and numerical reasoning – skills increasingly valued in modern scientific training.

7.3. Informal and outreach education

Beyond formal courses, this methodology holds significant potential for informal learning settings such as science museums, public workshops, and online educational platforms.

In these contexts, the focus shifts from formal assessment to scientific engagement and conceptual understanding. Interactive simulations can be embedded within public exhibits or virtual outreach modules, allowing diverse audiences to explore the effects of curved space through direct manipulation and visual storytelling.

Such experiences act as cognitive organisers, helping participants relate novel phenomena – such as the behaviour of vectors on curved surfaces – to their prior knowledge of motion, trajectories, or spatial relationships. For instance, a museum kiosk might invite visitors to 'guide a spacecraft across a curved planet' and observe how its direction changes, prompting intuitive questions about the nature of motion and geometry. Similarly, an online version of the simulation could be shared through open-access platforms, enabling self-directed exploration by hobbyists, educators, and students worldwide.

Such outreach activities help democratise access to advanced scientific ideas, demonstrating that concepts often reserved for specialised courses can be made approachable and meaningful for the general public. By connecting everyday experiences – such as GPS navigation or satellite motion – to the visual behaviour of vectors on curved surfaces, these implementations foster broader scientific appreciation and inspire curiosity about modern physics.

7.4. Resource availability and open access

To promote reproducibility and broad accessibility, all materials related to this study have been made openly available. In particular, an interactive simulation of parallel transport can be freely accessed online, enabling users to explore and visualise the concepts directly through a web interface: https://transporteparalelo-1.onrender.com/.

8. Conclusions

This study has presented and validated a novel teaching and approximation scheme for parallel transport on curved manifolds, illustrated through the canonical case of the two-dimensional sphere. By decomposing arbitrary curves into successive geodesics, the method achieves a balance between conceptual simplicity and computational rigour. Numerical analysis confirms that the accuracy improves systematically with the number of geodesic segments, thereby underscoring the robustness of the approach as an alternative to standard differential equation–based formulations.

Beyond its technical merits, the proposed framework demonstrates significant value in both theoretical and educational contexts. Theoretically, it offers an efficient tool for simulations in areas requiring repeated parallel transport, such as numerical relativity, computer graphics, and geometric analysis. Pedagogically, the construction provides an intuitive gateway into the geometry of curved spaces, enabling instructors to convey the essence of parallel transport without relying exclusively on abstract formalism.

The originality of this work lies in its integration of a rigorous geometric framework with a computational approach that enables systematic analysis and visualisation. By bridging geometric theory with numerical implementation, the method advances both the conceptual understanding of differential geometry and its practical exploration through computation. Future work may involve extending the scheme to higher-dimensional manifolds and more general curvature profiles, which would broaden its applicability and deepen its relevance. Thus, the contribution is not only methodological but also conceptual, fostering a productive dialogue between mathematical theory, numerical experimentation, and the pedagogy of modern physics.

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Author contributions

CRediT: **Juan Carlos Salazar Montenegro:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Resources, Software, Supervision, Validation, Visualization, Writing – review & editing; **Angela Viviana Gómez Azuero:** Conceptualization, Formal analysis, Investigation, Methodology, Software, Validation, Visualization, Writing – original draft, Writing – review & editing

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Data availability statement

The data supporting the findings of this study were generated by the authors through numerical simulations. Public sharing of the data is not applicable, as no external or experimental datasets were used.

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