Faddeev-Jackiw quantization of Proca's Electrodynamics on the null-plane

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The effective gauge invariance Lagrangian density which describe the Proca field is defined by:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M^2\left[A_\mu + \frac{1}{e}\partial_\mu\theta\right]^2, \tag{1}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The Lagrangian above is invariant by the following transformations,

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x) \qquad , \qquad \theta(x) \to \theta(x) - e\Lambda(x), \tag{2}$$

From (1) it is easy to write the first-order Lagrangian by introducing the momentum π^{μ} and p_{θ} with respect to the fields A_{μ} and θ, respectively,

$$\pi^{\mu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_{+} A_{\mu}\right)} = F^{\mu +} \quad , \quad p_{\theta} = \frac{\partial \mathcal{L}}{\partial\left(\partial_{+}\theta\right)} = \frac{M^{2}}{e}\left[A_{-} + \frac{1}{e}\partial_{-}\theta\right]. \tag{3}$$

from where, the starting Lagrangian density is written in first order as follow:

$$\mathcal{L}^{(0)} = \pi^{-}\partial_{+}A_{-} + \pi^{a}\partial_{+}A_{a} + p_{\theta}\partial_{+}\theta - \mathcal{H}^{(0)} \tag{4}$$

where the zero iterated symplectic potential has the following form:

$$\begin{aligned}\mathcal{H}^{(0)} &= \frac{1}{2}\pi^{-}\pi^{-}+\pi^{-}\partial_{-}A_{+}+\pi^{a}\partial_{a}A_{+}+\frac{1}{4}F_{ab}F_{ab}-eA_{+}p_{\theta}\\ &\quad -\frac{1}{2}M^{2}\left(A_{a}+\frac{1}{e}\partial_{a}\theta\right)\left(A_{a}+\frac{1}{e}\partial_{a}\theta\right).\end{aligned} \tag{5}$$

The initial set of symplectic variables is given by the set $\xi_k^{(0)} = (A_-, A_a, \theta, \pi^-, \pi^a, p_\theta, A_+)$. We have from (4) the canonical momenta,

$$\begin{aligned}K^{(0)}_{A_-} &= \pi^{-}, \quad K^{(0)}_{A_a}=\pi^{a}, \quad K^{(0)}_{\theta}=p_{\theta}\\ K^{(0)}_{\pi^-} &= 0, \quad K^{(0)}_{\pi^a}=0, \quad K^{(0)}_{p_\theta}=0, \quad K^{(0)}_{A_+}=0.\end{aligned} \tag{6}$$

The zero iterated symplectic two-form matrix is defined by,

$$M_{AB}^{(0)}(\mathbf{x},\mathbf{y}) = \frac{\delta K_B^{(0)}(\mathbf{y})}{\delta \xi_A^{(0)}(\mathbf{x})} - \frac{\delta K_A^{(0)}(\mathbf{x})}{\delta \xi_B^{(0)}(\mathbf{y})}. \tag{7}$$

with the components

$$M_{AB}^{(0)}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta_b^a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_b^a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta^3(\mathbf{x}-\mathbf{y}). \tag{8}$$

The symplectic matrix is singular and it has a zero mode

$$\tilde{v}^{1(0)}(\mathbf{x}) = \left(\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & v(\mathbf{x}) \end{array} \right), \quad (9)$$

where $v(\mathbf{x})$ is an arbitrary function. From this nontrivial zero-mode, we have the following constraint,

$$\begin{aligned} \Omega_1^{(0)} &= \int d^3xv(\mathbf{x}) \frac{\delta}{\delta\xi_{A_+}(\mathbf{x})} \int d^3y\mathcal{H}^{(0)}(y) \\ &= -\int d^3xv(\mathbf{x}) \left[\partial_-^x \pi^-(x) + \partial_a^x \pi^a(x) + ep_\theta(x) \right] \quad (10) \\ &= 0. \end{aligned}$$

The constraint is evaluated form (10) to be,

$$\Omega_1^{(0)} = \partial_-^x \pi^-(x) + \partial_a^x \pi^a(x) + ep_\theta(x) = 0. \quad (11)$$

The constraint (11) is introduced in the Lagrangian density, thus, the first iterated Lagrangian density is written as,

$$\mathcal{L}^{(1)} = \pi^{-}\partial_{+}A_{-} + \pi^{a}\partial_{+}A_{a} + p_{\theta}\partial_{+}\theta + \Omega_{3}^{(0)}\dot{\beta} - \mathcal{H}^{(1)} \tag{12}$$

where the first iterated symplectic potential is

$$\begin{aligned} \mathcal{H}^{(1)} &= \mathcal{H}^{(0)}_{\Omega_1^{(0)}=0} = \frac{1}{2}\pi^{-}\pi^{-} + \frac{1}{4}F_{ab}F_{ab} \\ &\quad -\frac{1}{2}M^{2}\left(A_{a} + \frac{1}{e}\partial_{a}\theta\right)\left(A_{a} + \frac{1}{e}\partial_{a}\theta\right). \end{aligned} \tag{13}$$

We enlarged the space defined by $\xi_{k}^{(1)} = (A_{-}, A_{a}, \theta, \pi^{-}, \pi^{a}, p_{\theta}, \beta)$.

The first iterated symplectic matrix is written as,

$$M_{AB}^{(1)}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta_b^a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -\partial_-^x \\ 0 & \delta_b^a & 0 & 0 & 0 & 0 & -\partial_a^x \\ 0 & 0 & 1 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & -\partial_-^x & -\partial_b^x & -e & 0 \end{pmatrix} \delta^3(\mathbf{x}-\mathbf{y}) \tag{14}$$

The modified symplectic matrix is again singular. The new constraint is identically zero.

In order to obtain a regular symplectic matrix a gauge fixing term must be added. We choose the null-plane gauge $\Theta = A_{-}(x) = 0$. We obtain the second iterative Lagrangian, i.e.:

$$\mathcal{L}^{(2)} = \pi^{-}\partial_{+}A_{-} + \pi^{a}\partial_{+}A_{a} + p_{\theta}\partial_{+}\theta + \Omega_{3}^{(0)}\dot{\beta} + \Theta\dot{\eta} - \mathcal{H}^{(2)} \quad (15)$$

where

$$\begin{aligned} \mathcal{H}^{(2)} &= \mathcal{H}^{(1)}_{\Omega_{i1}^{(0)},\Theta=0} = \frac{1}{2}\pi^{-}\pi^{-} + \frac{1}{4}F_{ab}F_{ab} \\ &\quad -\frac{1}{2}M^{2}\left(A_{a} + \frac{1}{e}\partial_{a}\theta\right)\left(A_{a} + \frac{1}{e}\partial_{a}\theta\right). \end{aligned} \quad (16)$$

The new set of symplectic variable is:
$\xi_k^{(2)} = (A_-, A_a, \theta, \pi^-, \pi^a, p_\theta, \beta, \eta)$ and from (15) we determine the second-iterated symplectic two-form matrix,

$$M_{AB}^{(2)}\left(\mathbf{x},\mathbf{y}\right) = \frac{\delta K_B^{(2)}\left(\mathbf{y}\right)}{\delta \xi_A^{(2)}\left(\mathbf{x}\right)} - \frac{\delta K_A^{(2)}\left(\mathbf{x}\right)}{\delta \xi_B^{(2)}\left(\mathbf{y}\right)}. \tag{17}$$

Since this matrix is not singular, we finally have the inverse matrix after a laborious calculation. From this relations and

$$\left\{ \xi^{(2)A}\left(\mathbf{x}\right), \xi^{(2)B}\left(\mathbf{y}\right) \right\} = \left[M_{AB}^{(2)}\left(\mathbf{x},\mathbf{y}\right) \right]^{-1}, \tag{18}$$

we immediately identify the generalized brackets as follow:

$$\left\{ A_a(x), \pi^-(y) \right\} = -\frac{\partial_a^x}{\partial_-^x}\delta^3(\mathbf{x}-\mathbf{y}),$$

$$\left\{ A_a(x), \pi^b(y) \right\} = \delta_b^a\delta^3(\mathbf{x}-\mathbf{y}),$$

$$\left\{ \theta(x), \pi^-(y) \right\} = \frac{e}{\partial_-^x}\delta^3(\mathbf{x}-\mathbf{y}),$$

$$\left\{ \theta(x), p_\theta(y) \right\} = \delta^3(\mathbf{x}-\mathbf{y}).$$

- The results give us the Dirac brackets of the theory.
- The structure of these constraints is very simple.
- The potential symplectic obtained at the final stage of iterations is exactly the Hamiltonian.