Pore network model on the SWCC and the hysteresis phenomena in unsaturated soils

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Abstract

This paper presents an alternative approach for modelling the soil water characteristic curve (SWCC) in unsaturated soils based on Monte Carlo simulations. The proposed approach is achieved in a two-phase process. First, the soil is simulated as a network of interconnected pores, in which each pore interacts with its closest neighbors. Second, a Monte Carlo simulation is used to estimate the average physical properties of the pore system. Results of the pore filling distribution in the soil, are shown as a function of the pressure difference. The proposed model reproduces the hysteresis curves caused by the hydraulic and mechanical drying and wet process. Finally, we examine the order parameter that attributes the theory to the first order transition observed in the hysteresis soil curve.

Keywords: Unsaturated soil, Pore network, SWCC, Hysteresis, Monte Carlo, Statistics, Ising model.

1 Introduction

Various geotechnical and engineering problems involve unsaturated soil mechanics. Practicing engineering have determinated that soil properties, such

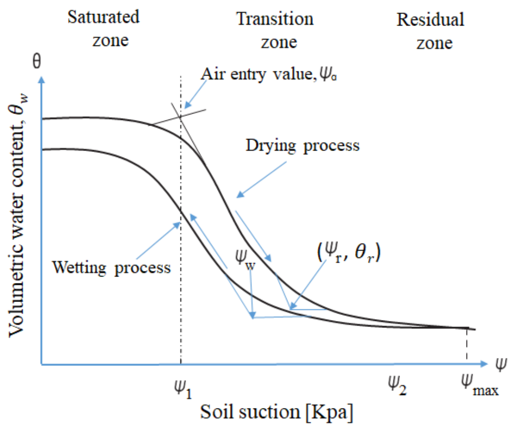

Fig. 1 Volumetric water content as function of the soil suction.

as, permeability, compression index, Young's Modulus, etc, which were previously assumed to be constant in saturated soil mechanics, have been found to vary substantially depending upon the degree of saturation of the soil [1]. It has been necessary to define a number of soil property functions to implement the practice of unsaturated soils. The behavior of unsaturated soils has been the subject of research for several decades. To date, there is enough field and laboratory data; however, all of this knowledge has been difficult to put into practice due to the excessive costs involved in property determination. Therefore, one of the challenges in the area of unsaturated soil mechanics is to propose viable methodologies to determine the functions that describe the behavior of these soils. The most used functions can be classifed into three general phenomena: Flow, stress and deformation [2]. Concretly, in these three categories are: the hydraulic conductivity, the function of resistance to shear stress, the void ratio function and the function of water content or characteristic curve (SWCC).

In recent years, the SWCC has become of great value in estimating unsaturated soil property functions [3–5]. The SWCC can be related to other properties describing the behavior of the soil, such as the unsaturated coefficient of permeability [6], the shear strenght [7], the coefficient of water volume change function, among others [8–10].

The SWCC relates the gravimetric water content, w, or volumetric water content, θ_w, to soil suction. Fig. 1 shows a typical drying and wetting SWCC. The point where air begins to enter the soil pores (during drainage) or where all air is expelled (during wetting) is known as the air entry value (AEV), ψ_a [11, 12]. As matric suction (ψ) is increased from zero to the AEV of the soil, the volumetric water content, θ_w, is nearly constant. Then the water content steadily decreases to the residual water content, θ_r, as matric suction increases beyond the ψ_a value. The residual water content is the water content at residual state, at which water phase is discontinuous. The soil suction corresponding to the residual water content is called the residual soil suction, ψ_r. The water entry value, ψ_w, on the wetting SWCC, is defined as the matric suction at which the water content of the soil starts to increase significantly during the wetting process [13].

Several approaches have been taken towards determination and the measurement of the total matric suction of the soil and the corresponding SWCC. Two major groups of models can be distinguished: laboratory and field methods, which vary in terms of cost, complexity and measurement range [14]. Some examples of these models include tensiomenters and contact filter paper techniques, pressure plate apparatus, thermal/electrical conductivity sensors for measuring matric suction and humidity measurement techniques. Also, a quite a number of models or equations have been proposed over the years to describe the highly nonlinear SWCC behavior [6, 15–22].

Among all SWCC characteristics, the hysteresis effect play one of the most crucial roles, as indicated by several researchers().The amount of liquid retained in a porous media is not uniquely defined by the value of the matric potential but is also dependent on the "history" of wetting and drying. The main reasons that result in hysteresis are the ink bottle, which refers to narrow points of connections between large cavities of adjoining pores, and the effect of the contact angle [23]. The relation $\theta_w = f(\psi)$ then includes the initial and main curves or boundary curves and an infinite number of scanning curves between these two main curves.()

Motivated by this joint progress from experimental and numerical advances in the SWCC determination, in this work the authors have studied the flow properties of water in unsaturated porous media, in a wetting and drying process. A correct prediction of the imbibition or drainage path in the soil is widely used for obtaining mechanical and hydraulic properties of soils, such deformation, permeability and flow [24].

Usually, pore network models have been applied widely for simulating the flow of water in clays [25]. In these models an accurate depiction of the clay as a network consisting of a set of interconnected pores is required. Because the capillary energy of a pore depends upon the saturation state of its connected neighbors, this problem should be treated as a many body interaction. In this context, the porous network presents a hysteresis behavior with respect to wetting history in the soil-water characteristic curves (SWCCs), which describing the relation between volumetric water content (θ) and suction (ψ) [26].

In order to determine the SWCC for clay soils, various experimental tests have been conducted [27–29]. However, these laboratory tests are generally time consuming and expensive [24]. Therefore, a need for new techniques has resulted in various empirical methods where parameters are fitted to experimental data [6, 15, 30, 31]. In an alternative approach and in order to eliminate the cost for laboratory experiments, some theoretical and physical models based on the pore size or in the interactions between pores have been presented [32, 33].

In above mentioned theoretical works, usually mean field approximation, similar to those addressed by statistical mechanics to simulate atomic and molecular systems, lattice gases [34], neural networks [35], among others, have been used [36–38]. However the not trivial behavior of the SWCC in clays has called into the questions the meaningfulness of obtain the retention curve using not only approximations for the porous systems. Additionally, the pressure imposed across distant boundaries in the clay sample in a drainage process with vertical drains, result in more complicated retention curves which usually are not included in the theoretical models [25].

To address this difficulty, the authors have explored the collective interactions of pores in clays under the effect of vertical drains. Using the Monte Carlo technique the drainage process in the clay has been simulated and analyzed. Monte Carlo simulation performs statistical analysis by building models of possible results by substituting a probability distribution for any factor that has inherent uncertainty [39]. It then allows to calculate average results related with the physical variables in the system, each time using a different set of random values from the probability functions. While other treatments include approximations to simulate the interaction between pores in a drainage process [25], to describe this complex process the Monte Carlo simulation calculated involve thousands or tens of thousands of recalculations of interaction before it is completed.

In this context, in this work the researchers have simulated a porous network, modified with vertical drains, using a mechanical statistical model. Because soils can be modeled as a complex network consisting of a large number of small individual pores interconnected through several narrow tunnels and the vertical drains as an external agent affecting all the system, the Ising model in two dimensions (2D) was used [39]. Additionally, the hysteretic behavior in the characteristic curve is explicated in this (2D) model, arguing that a phase transition is impossible if the interaction is short range, that is, if a given pore interacts with only a finite number of pores. In this way, a model to simulate the effect of the vertical drains in the saturation and imbibition process of soils is provided and a comparison between curves with and without vertical drains is illustrated.

The paper is organized as follows. Sec.II is a frame to explain the hysteresis presented in the soil characteristic curves. Sec.III describes the general formalism for addressing the interaction between a set of vertical drains and a soft soil. In Sec.IV a statistical model to evaluate the effect of drains in the characteristic curve in a consolidation process is presented. The numerical model is included in the porous description and developed in Sec.V. The numerical results using a Montecarlo simulation and retention curves of the system are presented in Sec.VI. Finally, Sec.VII summarizes the main conclusions.

2 Hysteresis in the soil-water characteristic curve

Before proceeding further, in this section the hysteretic behavior that predicts the theory in the SWCCs curves is described. Is well known that the soils hold more water at a given tension during desorption than during the absorption [40]. This history dependence behavior, in analogy with other similar physical process, is known as hysteresis. The hysteresis loop produced in the soil suction and water content relationship, when soils are cyclically wetted and dried, refers to the nonunique relation between the pressure head h, and the water content θ, in the SWCC curve [41].

During infiltration when the water content increases, also the pressure head increases, therefore the SWCC curve can be described by a unique function. However, after the reversal of the input direction in the drainage process, the pair (h,θ) does not follow the original path, and the SWCC curve must be described by a unique but different function.

Analyses to describe this hysteresis phenomenon usually adopt one of three approaches: theoretical based models, experimental-analytical models and numerical simulations descriptions.

To describe the retention behavior of the soils, in the theoretical approach two major groups of models can be distinguished: Independent [42–44] and dependent pore models [25, 45, 46]. Depending on wether pore drainage is assumed to be independent or dependent upon the state of neighboring pores. Usually, in these two models the soil matrix is divided into a set of pore domains characterized by wetting pore radii that control the filling of pores by water and drying pore radii that control their emptying [41].

The second approach usually begins with the scaling of the primary, secondary and higher order scanning curves from the main hysteresis curve [47–50] using different analytical expressions that describe the soil water characteristic curve SWCC. Some of the commonly used analytical models of SWCC in the literature are Brooks-Corey, Farrell-Larson, Fredlund-Xing, Gardner, McKee-Bumb, Kosugi, van Genuchten, Campbell, Russo, Tani, Omuto, and Dexter [20, 21, 51]. These models allow to determine in a satisfactory way some shape parameters known as hydraulic parameters from the curve fitting the water retention functions using experimental data. However, most of these methods for measuring the soil hydraulic parameters are time consuming and costly, therefore the generation of a data set suitable for calibrating the analytical model can require several months [41].

To address this difficulty, other treatments propose design transient experiments involving hysteresis and analyze the data with computation applications for obtaining information about hysteresis in the soil hydraulic properties [52]. Other proposals, exploits the experimental techniques with x-ray microtomography and the numerical simulations advantages to find the internal liquid

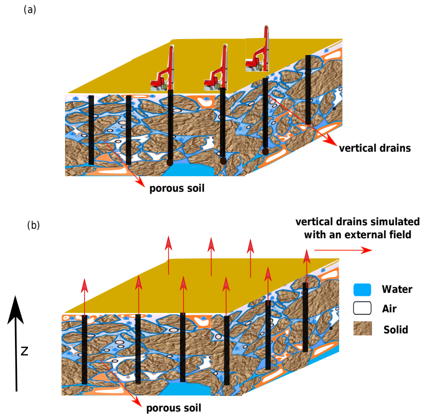

Fig. 2 Representation of a clay soil with a qualitative identification of the porous structure. (a) Schematic of prefabricated vertical drains installed by pushing a hollow steel mandrel over a soft soil. (b) Representation of the vertical drains with an external field acting in the direction $+z$ (out the plane). The soil is represented as a combination of solid, air and water.

distribution in the pores of a soil matrix [53–55]. However, numerical simulations are rarely large enough to represent realistic applications and are often limited to a relatively restrictive number of pores in the soil matrix.

In this paper, a theoretical and computational framework is explored. This involves collective interactions of pores in unsaturated soils. Without parametric fitting or analytical approximations, the behavior of the main fluid SWCC curves for a clay is obtained. To the present analysis the clay system is defined as a solid frame divided into a set of pore domains characterized by the state of pore filling.

Bulk domain behavior has been widely studied in magnetic material to explain the small groups of atoms in which all the electrons (tiny magnets) have the same magnetic orientation. One of the simplest and most important model in statistical physics to analyze the domain formation in magnetic systems, is the Ising model. In the absence of an external agent an analytical solution to the problem can be found, however under the effect of external fields in the system a numerical solution must be proposed. In this work, the authors propose to study the effect of vertical drains in porous soils using the Ising statistical model which predicts hysteresis in the magnetization variable in the case of magnetic materials. The main characteristics of the Ising model and the numerical proposal to approach that problem will be presented in the next section.

3 Model: porous systems

In this section, a model for describing hysteresis in the SWCC curve of clays will be constructed. This model includes the effect in the SWCC of a set of prefabricated vertical drains which are often used to accelerate the consolidation process (drainage process) on soft soils.

First, an element of soil of volume V is considered, Fig. 2. The total volume of the given sample can be expressed as $V = V_s + V_v$, where V_s represents the solid volume and V_v the void space. For simplicity, the void space is defined as a set of many interconnected pores on a two dimensional lattice. Furthermore, is assumed that pores can be filled only with water or air, therefore the void volume will have these two contributions $V_v = V_w + V_a$.

The aim is to find the energy that must be supplied to a pore to change its state from full to empty (or empty-full). To proceed further, a two-state description is adopted, characterizing the fill state of the pores σ as a binary variable. For convenience of notation, $\sigma = 1$ represent a pore with water, while $\sigma = -1$ a pore with air. In addition, the coordination number is examined, considering that each pore is connected with its neighbours through "tunnels", and there is a tendency towards the aggregation between pores with the same state of filling (domain formation).

To compute how a single pore state can change, also is requiered the effect of the vertical drains in the soils. This is included as an external agent affecting the whole system, in analogy with the external magnetic field in a magnetism system (arrows in Fig. 2(b)).

Suppose a pore of site i in the lattice whose initial state is completely full with water, the energy required to empty it ΔE_i is

$$\Delta E_i = \Delta E_{wa} + \Delta E_s + W_{wa}, \tag{1}$$

where ΔE_{wa}, is the change in surface energy that quantifies the disruption of inter-molecular bonds that occur when a surface is created (or destroyed). The surface energy of a pore depends strongly on the filling of all N connected pores of index j. Therefore, if the pore change of $\sigma_i = 1 \to -1$ and the neighbour pore state is $\sigma_j = -1$, a surface disappears with energy per unit of area γ_{wa_j} and (see Fig. 3(a)(b))

$$\Delta E_{wa_j} = -\gamma_{wa_j}, \tag{2}$$

if all the neighbour pores are included

$$\Delta E_{wa} = \sum_{j=1}^{N} \pm\gamma_{wa_j} a_{t,j}, \tag{3}$$

with γ_{wa_j} the surface energy between pairs of pores and $a_{t,j}$ the area of the tunnels that connect the pore i with its N closest neighbours, each one with

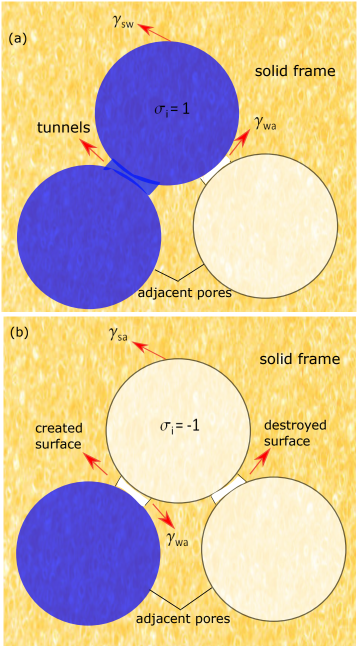

Fig. 3 Schematic representation of a identical porous assembly elements. (a) Single pore i initially filled with water ($\sigma_i = 1$) and their interactions with the external soil γ_{sw} and with the neighbour empty pores γ_{wa} (b) Representation of the surface destruction (creation) as a result of empty out the pore i ($\sigma_i = -1$).

area $a_{t,j}$. The term ΔE_s represents a change in the surface energy in the interface solid-water and solid-air. For instance, if the i pore is full with water $\sigma_i = 1$ and surrounded by solid material, destruction of the solid-water interface in the emptying process imply a change of the energy in the boundary interface from γ_{sw} to γ_{sa} (see Fig. 3(a)(b)), thus

$$\Delta E_s = (\gamma_{sa} - \gamma_{sw})A_p, \tag{4}$$

where A_p is the limiting solid area of the pores. Finally, the term W_{wa} in Eq.(1), represents the volume work asocciated with the pressure change (ψ_{ext}) in the pore i of volumen V_v in the emptying process. Therefore, if the pore state changes from $\sigma_i = 1 \rightarrow -1$, then the difference of energy will be given by

$$W_{wa} = \psi_{ext} V_p. \tag{5}$$

A similar argument shows that, for the opposite process ($\sigma_i = -1 \rightarrow \sigma_i = 1$), the signs in Eq.(2), Eq.(4) and Eq.(5) are flipped. In the following section the physical model and their main implications will be used to illustrate how this theory allows to predice hysteretic in the SWCC curves of the clay.

4 Ising model in porous systems

Equipped with the analysis in the previous section, a model describing the main characteristics and the qualitative behavior of the SWCC curve is constructed, based on the Ising-spin description [39]. The ising model, is one of the simplest, yet one of the most important, in statistical physics [56]. It was introduced in 1920 by Lenz as a model of a ferromagnet to explain the domain formation in magnetism [57]. In this paper, the process suggested in some recent studies, to model a pore network in the soil as an equivalent set of small magnets is implemented [25]. Specifically, a Ising variable σ_i ($\sigma_i = \pm 1$) is placed at site i on a bidemensional layer k of the soil, to describe the pore state filling in a site i of a lattice with N pores. The Hamiltonian (total energy) of the porous material is then proposed in the following way: First, is assumed that there exists nearest-neighbour interactions in each layer of the porous soil. Additionally, a tendency toward agregation between pores with the same filling state is taken into account by the 'ferromagnetic' nearest-neighbor interactions on the layer. According to Eq.(1), the dimensionless Hamiltonian describing such intra-layer coupling or interactions between the pores then reads

$$H(\sigma_i) = -\beta \sum_{\langle i,j \rangle} J_j \sigma_i \sigma_j - \beta \psi^{'} \sum_i \sigma_i, \tag{6}$$

where J_j is the interaction energy between the pore i and the j neighbouring pores in the lattice

$$J_j = \frac{\gamma_{wa_j} a_{t,j}}{2}. \tag{7}$$

The pore state of the neighbours is characterized with σ_j, which can take values $\sigma_j = \pm 1$.

Now, the strength of $\psi^{'}$ is used to simulate the external pressures in the soil, in analogy with an uniform and external field in a Ising magnetic chain

$$\psi^{'} = \frac{\Delta E_s - \psi_{ext} V_p}{2}. \tag{8}$$

Here the effects of the vertical drains has been included in the expression ψ_{ext}. This assumption is indeed consistent with the observed phenomenon that a drain causes a flow of the fluid in the upward direction as a result of a pressure difference. Therefore, the term ψ_{ext} acts as a force per unit of area which benefits the drainage due to the reduction of the $\psi^{'}$ value.

The β factor in Eq.(6) is the reciprocal of the thermodynamic temperature for systems with thermal interactions. Because, in this article energy transfer in heat form has not been considered, a statistical interpretation of β is used. In this framework, a large ensemble of copies of porous sample are considered, each copy with a distinct distribution of filling states. Copies with the same energy form a microstate. In this context, β is a property of the system that defines the variation in the number of accessible states due to a change in energy, where the number of accessible states represents the diverse and complex combinations of pores holding a given filling state. Aditionally, β preserves certain conditions that define the system at a macro level. For example, if the β value increases, the systems tends to ordered configurations which indicate lower energy.

In that follows, the total interaction energy $J = \sum_{j=1}^{N} J_j$, the external pressure ψ' and β will remain as control parameters of the system. Depending on the values of the control parameters, the system may exhibit ordered or disordered states together with the intermediate critical states.

5 Monte Carlo simulation

One of the main properties of the Ising model, presented in the last section, is related to the possibility for calculate phase transitions in the interest system. During a phase transition, certain properties change, often discontinuously, as a result of the change of some external conditions, such as temperature, pressure, among other. In this case, as a result of an external pressure due to the vertical drains inserted in the soil.

To probe the emergent phases of the Ising model, in this section a Monte Carlo simulation has been carried out under fixed boundary conditions in the porous lattice and with an external pressure applied. Conceptually it can be seen as a random walk through state space of the system. This means that it is important to average over large numbers of iterations in order to calculate averages over the interest variables. Aditionally, C++ language has been used to generate the random numbers, as these are essential elements of all Monte-Carlo methods.

At the start of the simulation, an initial configuration for the porous system has been chosen. Usually, two options are available: 1) An initial configuration in which all the pores are full with liquid (or full with air) and 2) a configuration in which each pore is independently full with air or water at random. For our purposes, the first option has been selected. The next step is to choose a pore at random, and change its state only if the result lowers the energy. This process must be repeated until a stable configuration is obtained. The transition rates in the system are given by

$$\begin{aligned} Rate[\sigma_i = 1 \rightarrow \sigma_i = -1] &= \Gamma, if \Delta E_i \leq 0; \\ &= 0, otherwise; \end{aligned} \tag{9}$$

where ΔE_i is the change of energy of the system as a result of the interactions between the nearest neighbour pores. In order to determine when equilibrium is reached, the average energy per pore $\langle E \rangle$ and the filling states for all the pores $\langle \sigma \rangle$ have been calculated, once the values of the running averages have converged to within a small bound, the system is considered to have reached equilibrium.

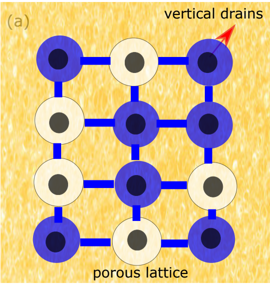

Fig. 4 Top view of a piece of soil. (a) Empty (white) and full spheres (blue) represent the pores and the tunnels are shown as rectangles. Above each pore have been considered a vertical drain (small black circles).

After experimentation and visual inspection, a equilibrated average energy and pore filling was found. Now, to explore the phase transitions in the model, concretely the hysteresis behavior in the SWCC curve the following procedure was performed: start with a sufficiently large negative applied external pressure $\psi^{'}$, so that in the stable configuration all the pores have state $\sigma = 1$, and increase the pressure slowly. At some value of $\psi^{'}$ the local pressure per pore will become positive, and this pore will change its state. This changes the effective pressure at the neighbour pores, and some of them may change state, causing an avalanche of changes. The average in the total state of filling the pores $\langle\sigma\rangle$ was determined when the avalanche had stopped. Then the applied pressure was increased a bit more, and the filling states for the pores in the stable state were determined again. The process was continued until all the pores were in the same state of filling. This procedure allows to obtain the lower half of the hysteresis loop $\langle\sigma_u\rangle$. The upper half of the hysteresis loop is obtained when the external pressure $\psi^{'}$ is decreased. This is related to the lower half of the loop $\langle\sigma_l\rangle$ by symmetry

$$\langle\sigma_u\rangle(\psi^{'}) = -\langle\sigma_l\rangle(-\psi^{'}) \tag{10}$$

An important feature of the above dynamics is that if the system starts with any stable configuration, and the external pressure is increased allowing the system relaxation, therefore no pores change their state more than once. Furthermore, the final stable configuration is the same regardless of the order in which unstable pores are changed of state.

6 Results

The simulation algorithm involves the construction of a random two-dimensional lattice, having $N \times N$ sites (which represents the pores) such that each site has exactly four neighbours, Fig.4. The precise algorithm that was used is as follows: label each site (each pore) with a pair of index (i, j). Connect the site (i, j) to site $(i, j + 1)$, $(i, j - 1)$, $(i + 1, j)$, $(i - 1, j)$ for all sites. Additionally, periodic boundary conditions were considered. In this simulation $N = 10^2$ and $NMC = 10^4$ Monte Carlo steps were used.

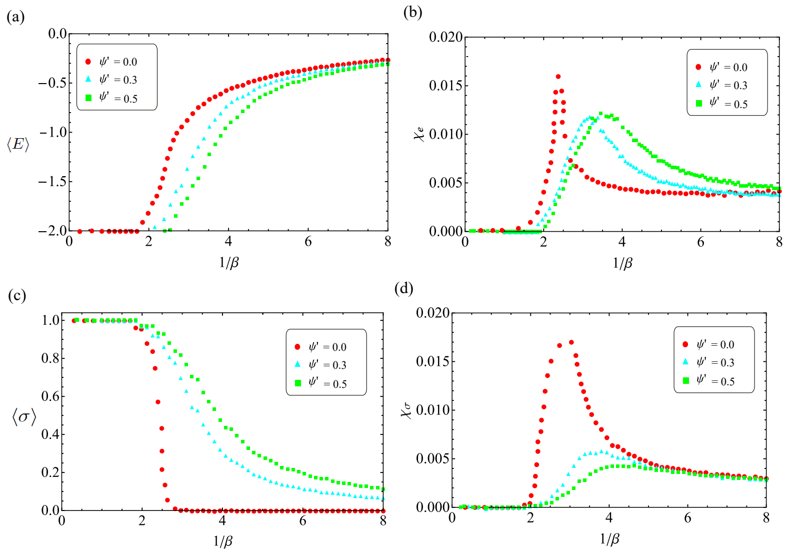

Fig. 5 Order parameters (a) $\langle E \rangle$ and (c) $\langle \sigma \rangle$ and their fluctuations (b) χ_e and (d) χ_σ. The peaks in (b) and (d) indicate the values of $1/\beta$ at which a phase transition in the mean energy and the filled pore state occurs in the system depending on the external pressure.

To analyse the critical behavior of the system, some order parameters were calculated. An order parameter is a measure of the degree of order across the boundaries in a phase transition system.

Concretely, two order parameters were considered: The average energy of interaction per pore $\langle E \rangle$ and the filling states for all the pores $\langle \sigma \rangle$. In Fig.(5(a)) the results of $\langle E \rangle$ are presented as a function of the inverse β factor, with three values of the external pressure $\psi^{'}$: $0, 0.3, 0.5$. The results for $\langle \sigma \rangle$ as a function of $1/\beta$ are plotted in Fig.(5(c)). In this graph it can be observed that if the difference between the pressure generated by the drains (upward) and the pressure associated to the effect of the surface soil-water and soil-air (downward) is $\psi^{'} = 0$, the system fluctuated between the positive (pores filled with water) and negative state (pores filled with air) of filling of the pores, depending on the $1/\beta$ value. It was found that for $\psi^{'} > 0$, there is an effect of shifting in the curves to the right, implying that the phase transition occurs at larger values of $1/\beta$. In other words, the system needs so much energy for a phase transition to occur because this biased behaviour encourages the pores to change their filling state slowly.

According to this model, the term ΔE_s favors the pores to be full of water, therefore the effect of the drains is counteracted by that term as can be observed in Eq.(8). In the case where ψ_{ext} is increased, ψ' tends to zero, facilitating drainage of pores as seen in Fig.(5)

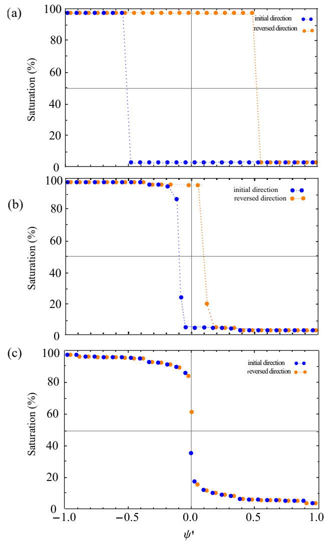

Fig. 6 Plot of the fraction between the volume of fluid that contains the sample and the volume that is admissible to be filled (saturation), against external pressure $\psi^{'}$, for different $1/\beta$ values.(a) $1/\beta = 1.5$, (b) $1/\beta = 1.9$. (c) $1/\beta = 2.5$. Notice the hysteresis loop formed by changing the external pressure, associated to the vertical drains, from -1 to 1.

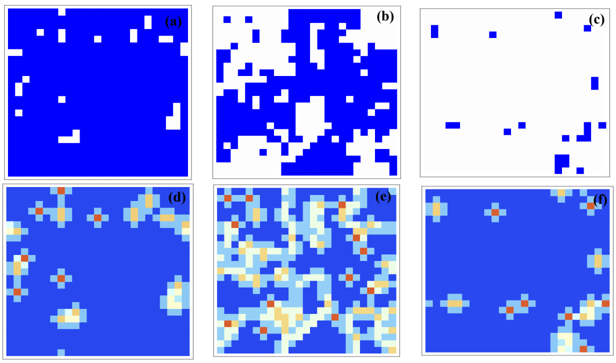

Fig. 7 Three random microstates of a $10^2 \times 10^2$ porous lattice with periodic boundary conditions corresponding to an equilibrium macrostate with $1/\beta = 2.5$ for different values of external pressure. (a) $\psi^{'} = -0.48$. (b) $\psi^{'} = 0$. (c) $\psi^{'} = 0.48$. The energy per pore in each one of the configurations is shown in (d) $\psi^{'} = -0.48$. (e) $\psi^{'} = 0$. (f) $\psi^{'} = 0.48$.

Fluctuations in the mean energy χ_e and filled states χ_σ are displayed in In Fig.(5(b)), In Fig.(5(d)) the peak position in χ_e and χ_σ curves also change with increasing $\psi^{'}$, which evidences a shifting in the value of the required energy in a phase transition depending on the pressure generated by the drains.

The non-equilibrium response of the Ising lattice at constant energy $(1/\beta)$, or equivalently, the saturation varying the pressure ψ' (SWCC curve) was investigated in Fig.(6). At low energies ($1/\beta = 1.5$) a rectangular loop is obtained which shows that the process of filling and emptying the pores by changing the external pressure in the system go through different paths in the saturation curve, Fig(6(a)). If the energy is increased, the area of the hysteresis loops decreases Fig(6(b)), until a limit value in which the hysteresis loop disappears Fig(6(c)). The observed hysteresis loop is due to a first-order phase transition from a meta-stable state to the true equilibrium occurred at different value of the pressure ψ'.

To gain further insight into the behavior of these statistical quantities, the spatial pattern of pores in the lattice corresponding to equilibrium micro-states with $1/\beta = 2.5$ for three values of the pressure $\psi^{'}$ are plotted in Fig.(7)(a),(b),(c). At a pressure $\psi^{'} = -0.48$ the system tends to stay with the pores filled with water, when the pressure associated to the drains is equal to the pressure in the soil material the pores are randomly filled as expected. When the $\psi^{'} = 0.48$ becomes positive the pores tends to a state filled with air, as we expect in a drainage process. The Fig.(7)(d),(e),(f) shows the energy per pore for a micro-state with $1/\beta = 2.5$ and for different values of the external pressure.

These spatial patterns give an intuitive picture of the behaviour of the saturation Fig.(6), χ_e and χ_σ Fig.(5)(b)(d) as the energy ($1/\beta$) is varied. At the critical point, there is a bifurcation leading to two equally probable equilibrium macro-states (corresponding to positive (1) or negative (-1) saturation). The system here is stuck between these two possibilities and fluctuates from one to the other. This explains why χ_e and χ_σ suddenly peak around this point.

7 Conclusions

In this research, an alternative approach for modeling consolidation acceleration with vertical drains in unsaturated soils was developed. The main goal was: (1) To construct a theoretical model for estimating the SWCC curve including the vertical drains effect. (2) To perform a numerical simulation to estimate the physical average properties of the soil. To this end, two steps were taken. In the first step, the soil was simulated as a network of innterconected pores. In the second step, based on the interactions between pores, a statistical model, called the Ising model, was contructed through which the filling pore state and vertical drains effect in the soils were studied. The Ising model was introduced in 1920 as a model of ferromagnetic systems. The model consists on the representation of a magnetic system as a discrete variable arranged in a lattice. This model includes two types of interactions: the interaction term between neighboring (magnets) and the external field term which represents external effects in the system. In this work the Ising model was implemented considering an analogy between the tendency toward aggregation between pores with the same filling states (called the ferromagnetic interaction between pores) and the interaction between a set of small magnets. The vertical drains were included in the model like the external field in the Ising model, this term tends to reduce the internal pressure in the system. To probe the emergent phases of the model, a Monte Carlo simulation was presented, this provide a rich variety of measured quantities in the consolidation process of a soft soil, such as: The mean energy, the filling state of the pores and the SWCC curve. From a theoretical perspective, these physical quantities, represent the order parameters , because range between one value m_1 in one phase and m_2 in the other phase. Usually, the order parameters arise from symmetry breaking. When this happens, it is necessary to introduce one or more extra variables to describe the state of the system. In the case of the consolidation process, the net suction was provided.

Aditionally, the proposed model allow to reproduce the hysteresis behaviour in the SWCC curve for a consolidation process with vertical drains. The presented results evidence the required energy to obtain phase transitions in the system and the dependence on these values with the difference between the pressure generated by the drains and the pressure due to the solid material in the soil. The limit value of energy to obtain hysteresis in the system was presented, and the results show that the hysteresis in the SWCC curve depends on the global effect of the drains in the soil. Further work would be considered in order to develop a more complex system, i.e. including pores partially filled in several percentages. This, could offer a more realistic representation of the soils.

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