Hamilton-Jacobi formulation for topological gauge theories

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We develop a Hamilton–Jacobi formulation for topological gauge theories, focusing on the Chern–Simons and Maxwell–Chern–Simons models. A complete constraint analysis is performed, allowing the identification of involutive and non-involutive Hamiltonians, the construction of the associated characteristic equations, and the determination of the generators of gauge transformations. Particular emphasis is placed on the incorporation of gauge-fixing conditions into the Hamilton-Jacobi framework and on their role in the integrability structure of the theory. We show that this implementation preserves the integrability of the system and leads to a consistent generalized bracket structure, in full agreement with the results obtained through the Dirac formalism. These results provide a unified and self-consistent description of constrained gauge systems within the Hamilton–Jacobi approach.

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1. Introduction

It is well established that non-Abelian gauge theories formulated in (2+1) spacetime dimensions are super-renormalizable but suffer from severe infrared divergences. A consistent mechanism to circumvent these divergences consists in introducing a topological Chern–Simons (CS) term into the gauge-invariant action, thereby constructing what is known as a topologically massive gauge theory.[1] Such models, originally proposed by Deser, Jackiw, and Templeton,[2,3] arise from the addition of a CS term to the three-dimensional Maxwell action, leading to the well-known Maxwell–Chern–Simons (MCS) theory. The inclusion of the CS term endows the gauge field with a topological mass while preserving gauge invariance. At the quantum level, the induced topological mass acts as an infrared regulator, thereby removing the pathological divergences in the low-energy regime.

Following Dirac's pioneering work on constrained systems,[4] this formalism has become an essential tool for studying gauge-invariant topologically massive theories

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through canonical analysis. Using Dirac's method, the Chern–Simons and Pontryagin theories have been systematically examined, leading to the identification of their extended Hamiltonians, extended actions, constraint algebras, gauge transformations, and physical degrees of freedom.[5] The Maxwell–Chern–Simons (MCS) theory has also been quantized using the Dirac bracket approach and the Schwinger action principle.[6] Moreover, the first-order formulation of Maxwell and U(1) gauge theories, where a gauge-invariant mass term naturally emerges, has been analyzed via the Dirac procedure, revealing the gauge transformations that preserve the action and enabling the extension to non-Abelian generalizations.[7]

The Canonical Formalism of Dirac has historically been the standard for the quantization of constrained gauge systems. However, the Faddeev-Jackiw Symplectic Quantization method,[8] formalized by Barcelos-Neto and Wotzasek,[9,10] offers an alternative methodology that is intrinsically geometric and often more efficient. The key advantage of the FJ approach lies in its ability to handle both first- and second-class constraints in a unified manner, leading to the direct derivation of the Dirac brackets through the inversion of a reduced symplectic matrix. This procedural efficiency has proven invaluable, having been successfully applied to the non-Abelian Chern-Simons theory[11] and more complex systems, such as the higher-derivative CS theory,[12] consistently confirming the equivalence and coherence of the results with the traditional Dirac canonical quantization.

An approach based on the Hamilton–Jacobi (HJ) formalism was developed by Güler[13] to study singular first-order systems. This method employs Carathéodory's equivalent Lagrangians approach to the calculus of variations.[14] For such systems, the stationary action condition reduces to a set of Hamilton–Jacobi partial differential equations (HJPDEs), also referred to as Hamiltonians, which must satisfy an integrability condition (IC). The HJ formalism has been successfully applied to several models, including the Schwinger model,[15] the Proca model,[16] Teleparallel Gravity,[17,18] and Berezin systems.[19] Furthermore, the formalism has been extended to analyze systems described by higher-order Lagrangians.[20,21] The study of first-order actions led to the construction of generalized brackets,[22,23] showing that for any Hamiltonian that does not satisfy the integrability condition, such brackets can be consistently defined. A similar constraint analysis applied to the two-dimensional BF model[24] revealed a rich algebraic structure. In the case of the topologically massive Yang–Mills theory, the generators of Lagrangian gauge transformations were also found to be directly related to the Hamiltonians.[25]

Although the Hamilton–Jacobi formalism has been applied to constrained systems and gauge theories in various contexts, certain aspects remain insufficiently explored. In particular, the role of gauge fixing within this framework has not been fully clarified. Although gauge conditions have been considered in previous Hamilton–Jacobi analyses, their incorporation as additional Hamilton–Jacobi partial differential equations within the fundamental differential and their interpretation as part of the integrability structure of topological gauge field theories deserve further investigation.

In this work, we address this issue by introducing gauge fixing conditions directly into the Hamilton–Jacobi scheme and analyzing their impact on the integrability conditions and the resulting generalized bracket structure. We show that this procedure not only preserves the internal consistency of the formalism but also ensures full agreement with the results obtained through the Dirac method. This establishes a unified and self-consistent description of constrained gauge systems within the Hamilton–Jacobi approach.

More specifically, we perform a constraint analysis of the Chern–Simons (CS) and Maxwell–Chern–Simons (MCS) theories within the Hamilton–Jacobi formalism. The main objective is to identify the complete set of involutive and non-involutive Hamiltonians, construct the corresponding characteristic equations, determine the generators of the gauge transformations, and implement the appropriate gauge fixing conditions.

Previous developments of the Hamilton-Jacobi formalism have established important connections between integrability and gauge symmetries in constrained systems. In particular, Bertin, Pimentel, and Valcarcel[26] showed that involutive Hamiltonians generate canonical transformations and provide the fundamental ingredients for constructing the generators of gauge symmetries. More recently, Romero Hernández *et al.*[27] investigated finite-dimensional constrained systems containing involutive and non-involutive Hamiltonians and employed gauge conditions to study the correspondence between generalized brackets and Dirac brackets.

In the present work, we extend this discussion to topological gauge field theories and adopt a different perspective regarding gauge fixing. Rather than introducing gauge conditions solely as auxiliary constraints, we incorporate them as additional HJPDE within the fundamental differential itself. Since these conditions are not initially associated with independent variables of the theory, the procedure requires an enlargement of the parameter space by assigning an independent parameter to each additional Hamiltonian. Consequently, gauge fixing becomes part of the integrability structure of the Hamilton-Jacobi formalism, leading to an enlarged non-involutive system from which the arbitrary parameters are completely determined through the integrability conditions. This construction naturally yields generalized brackets equivalent to the corresponding Dirac brackets.

The paper is organized as follows: in Section II, we provide a brief overview of the Hamilton–Jacobi formalism. Section III is devoted to the application of this method to the CS theory, while Section IV presents its extension to the MCS case. Finally, the results and conclusions are discussed in Section V.

2. The Hamilton-Jacobi Formalism

We consider a system described by a Lagrangian of the form $L = L(x^i, \dot{x}^i, t)$ with $i = 1, 2, \ldots, N$ and whose Hessian matrix is singular and has rank P. It is convenient to split the configuration variables as $x^i = (x^a, x^z)$, with $a = 1, 2, \ldots, P$ and $z = 1, 2, \ldots, R$, such that $P + R = N$. The variables x^a correspond to the nonsingular sector of the Hessian matrix, whereas x^z belong to its singular sector. Within Carathéodory's equivalent Lagrangian method in the calculus of variations, the necessary and sufficient condition for extremizing the action reads $p_0 + p_a\dot{x}^a + p_z\dot{x}^z - L = 0$, where the canonical momenta are defined as $p_a = \frac{\partial S}{\partial x^a}$, $p_z = \frac{\partial S}{\partial x^z}$ and $p_0 = \frac{\partial S}{\partial t}$. The singularity of the Hessian matrix implies the existence of R canonical constraints $H_z \equiv p_z - \frac{\partial L}{\partial \dot{x}^z} = 0$. This allows us to introduce the canonical Hamiltonian $H_0 = p_a\dot{x}^a + p_z\dot{x}^z - L$, leading naturally to a set of $R+1$ first-order HJPDE,

$$H_\alpha \equiv p_\alpha - \frac{\partial L}{\partial \dot{t}^\alpha} = 0, \qquad \alpha = 0, 1, 2, \ldots, R, \tag{1}$$

where $t^\alpha = (t, x^z)$ and the quantities H_α are referred to as the Hamiltonians of the system.

By means of Cauchy's method, the set of equations (1) can be related to a system of total differential equations, known as the characteristic equations (CE),

$$dx^a = \frac{\partial H_\alpha}{\partial p_a} dt^\alpha, \quad dp_a = -\frac{\partial H_\alpha}{\partial x^a} dt^\alpha, \quad dS = p_a dx^a - H_\alpha dt^\alpha. \tag{2}$$

Solutions of the first pair of equations define curves depending on $R+1$ parameters t^α in the phase space spanned by the conjugate variables (x^a, p_a). From Eq. (2), and regarding the parameters t^α as independent, the evolution of any phase-space function

$$F = F(x^a, t^\alpha, p_a, p_\alpha) \tag{3}$$

is governed by the fundamental differential

$$dF = \{F, H_\alpha\} dt^\alpha, \tag{4}$$

where the Poisson brackets (PB) are defined on the extended phase space $(x^a, t^\alpha, p_a, p_\alpha)$. Hence, the Hamiltonians H_α act as generators of the dynamical evolution of the system.

The existence of a complete solution of the HJPDE, together with the independence of the parameters t^α, is guaranteed by the integrability condition (IC), which can be expressed as,

$$\{H_\alpha, H_\beta\} = C_{\alpha\beta}^{\ \ \gamma} H_\gamma. \tag{5}$$

That is, the Hamiltonians must close a Lie algebra under the Poisson brackets. Hamiltonians satisfying this condition are called involutive Hamiltonians. The presence of non-involutive Hamiltonians indicates that the system of HJPDE is incomplete or that the parameters are not linearly independent. In such cases, the equivalent condition $dH_\alpha = 0$ may generate new constraints and reveal possible dependencies among the parameters. The Hamilton–Jacobi constraint analysis thus consists of identifying a complete set of involutive Hamiltonians.

If, after completing the set of HJPDE, a subset of non-involutive Hamiltonians H'_x remains, one may construct the matrix with elements $M_{xy} \equiv \{H_x, H_y\}$. The dynamics is then redefined by eliminating the parameters associated with these Hamiltonians through the introduction of generalized brackets (GB). If the matrix M is singular of rank $K \leq R$, one can identify a regular submatrix $M_{\bar{a}\bar{b}}$, with $\bar{a} = 1, 2, \ldots, K$, and define the GB as[26]

$$\{A, B\}^* = \{A, B\} - \{A, H_{\bar{a}}\}(M_{\bar{a}\bar{b}})^{-1}\{H_{\bar{b}}, B\}. \tag{6}$$

The fundamental differential can then be written as

$$dF = \{F, H_{\bar{\alpha}}\}^* dt^{\bar{\alpha}}, \qquad \bar{\alpha} = 0, K+1, \ldots, R. \tag{7}$$

After this procedure, the Hamiltonians $H_{\bar{\alpha}}$ constitute the complete set of involutive Hamiltonians governing the dynamics of the system, with the Poisson brackets replaced by the GB.

The dynamical evolution described by the remaining arbitrary parameters can be interpreted as canonical transformations generated by the involutive Hamiltonians. In order to establish the relation between canonical and gauge transformations, we restrict the analysis to fixed times, $dt^0 = 0$. Under this condition, the variation of any phase-space variable γ^I is given by

$$\delta\gamma^I = \{\gamma^I, H_z\}^* \, \delta t^z. \tag{8}$$

Although the involutivity condition implies $\{H_x, H_y\}^* = C^z_{xy} H_z$, the IC generally yields $\{H_x, H_y\}^* = C^0_{xy} H_0 + C^z_{xy} H_z$. Since the condition $C^0_{xy} = 0$ is rarely satisfied, and the constraints $H_\alpha = 0$ restrict the phase space, one can define the generator of gauge transformations as[26]

$$G^{\text{can}} \equiv H_z \, \delta t^z. \tag{9}$$

where $\delta\gamma^I = \{\gamma^I, G^{\text{can}}\}^*$.

3. Hamilton-Jacobi Analysis for Pure $U(1)$ Chern-Simons Theory

The Lagrange density of the pure $U(1)$ Chern-Simons theory in $(2+1)$ dimensions is given by

$$\mathscr{L}_{\text{CS}} = \frac{k}{8\pi} \varepsilon^{\alpha\beta\gamma} F_{\alpha\beta} A_\gamma, \tag{10}$$

where k is a dimensionless coupling constant and A_γ is the gauge potential where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. In this article, we will use the convention $\varepsilon^{012} = 1$ and the Minkowski metric $g_{\alpha\beta} = (1, -1, -1)$. It can be verified that (10) is invariant under the gauge transformation up to a boundary term. The corresponding field equations are

$$\frac{k}{4\pi} \varepsilon^{\alpha\beta\gamma} F_{\beta\gamma} = 0, \tag{11}$$

3.1. *Non-involutive HJPDE*

Now, to begin with the Hamilton-Jacobi (HJ) formalism, we compute de momenta Π^μ conjugate to A_ν

$$\Pi^\mu \equiv \frac{\mathscr{L}_{\rm CS}}{\partial(\partial_0 A_\mu)} = \frac{k}{4\pi}\varepsilon^{0\mu\nu}A_\nu. \tag{12}$$

The expression above do not depend on any velocities $\partial_0 A_\mu$, which implies that the system is singular. The canonical Hamiltonian is given by

$$H_{\rm CS} = -\frac{k}{2\pi}\int d^2x A_0 \varepsilon^{0ij}\partial_i A_j, \tag{13}$$

Let us define $p \equiv \partial_0 S$, then in the context of the HJ formalism, we have the initial set of Hamilton-Jacobi Partial Differential Equations (HJPDE)

$$H_0 \equiv p - \frac{k}{2\pi}A_0\varepsilon^{0ij}\left(\partial_i^x A_j\right), \qquad \to x^0, \tag{14a}$$

$$H_1 \equiv \Pi^0, \qquad \to \tau_1, \tag{14b}$$

$$H^i{}_2 \equiv -\Pi^i + \frac{k}{4\pi}\varepsilon^{0ij}A_j, \qquad \to \tau_{i,2}. \tag{14c}$$

with $i = 1, 2$. From these densities we identify the following parameters of the theory: the first Hamiltonian H_0 is associated with the time parameter $x^0 = t$. The Hamiltonians H_a, with $a = 1, 2, 3$, which arose from the non-invertible momenta Π^μ are related to the parameters τ_a. The fundamental Poisson brackets (PB) are given by

$$\{A_\mu(x), \Pi^\nu(y)\} = \delta_\mu{}^\nu \delta^2(\mathbf{x}-\mathbf{y}). \tag{15}$$

Given the variables of the theory, and by defining the PB we build the fundamental differential wich characterizes the evolution of any function of the phase space. It is expressed as

$$dF(x) = \int d^2y \{F(x), H_\rho(y)\} d\tau_\rho(y), \tag{16}$$

with $\{\sigma\} = \{0, ..., 3\}$ and $\tau_0(y) \equiv t$. The relation (16) establishes the dynamic evolution of any function defined in the phase space. Now, it is necessary to verify the integrability conditions (IC), in order to ensure that the set HJPDE (14) is complete. The IC on the equations (14) imply that the following condition

$$dH_\sigma(x) = \int d^2y \{H_\sigma(x), H_\rho(y)\} d\tau_\rho(y) = 0, \tag{17}$$

be imposed on the system. The condition for H_1 gives,

$$dH\ _1 = \frac{k}{2\pi}\varepsilon^{0ij}\left[\partial_i A_j\right] dt = 0. \tag{18}$$

Since $H^0{}_1$ is not in involution, IC give new HJPDE:

$$H_4 \equiv \varepsilon^{0ij}\partial_i A_j = 0, \tag{19}$$

and the IC have to be tested with this too. Hence, we apply

$$dH_4 = dt\varepsilon^{0ik}\partial_i\partial_k A_0 = 0, \tag{20}$$

which is automatically satisfied. Then, the integrability of H_4 will not give any new Hamiltonian density. Now, following the HJ formalism, we need to test the IC for ${H^i}_2$. This is, we need to impost $d{H^i}_2 = 0$ and see if we obtain identities or new HJPDE. Thus, we obtain

$$d{H^i}_2 = \frac{k}{2\pi}\varepsilon^{0ik}\left[dt\partial_k A_0 + d\tau_{k2}\right] = 0, \tag{21}$$

which establishes a relationship between $d\tau_{k2}$ and dt as follows

$$d\tau_{j2} = -dt\partial_j A_0. \tag{22}$$

Therefore, we end up with the following complete set of HJPDE

$$H_0 = p - \frac{k}{2\pi}A_0\varepsilon^{0ij}\partial_i A_j, \tag{23a}$$

$$H_1 = \Pi^0, \tag{23b}$$

$${H^i}_2 = -\Pi^i + \frac{k}{4\pi}\varepsilon^{0ij}A_j, \tag{23c}$$

$$H_4 = \varepsilon^{0ij}\partial_i A_j. \tag{23d}$$

At this stage, we will separate de HJPDE (23) others than H_0 as involutive or non-involutive. The separation is determine by the PB between the complete set of HJPDE, this allows us reduce the phase space of the theory. We notice that H_1 is a involutive Hamiltonian density, however, from the brackets associated with the densities ${H^i}_2$ and H_4 we determine that the following matrix $\Phi_{a,b}(x,y) \equiv \{H_a(x), H_b(y)\}$ with $H_1 \equiv {H^1}_2$, $H_2 \equiv {H^2}_2$ and $H_3 \equiv H_4$,

$$\Phi(\mathbf{x},\mathbf{y}) = \begin{pmatrix} -\frac{k}{2\pi}\varepsilon^{0ij} & \varepsilon^{0ik}\partial^x_k \\ \varepsilon^{0jk}\partial^x_k & 0 \end{pmatrix}\delta^2(\mathbf{x}-\mathbf{y}), \tag{24}$$

is singular and of rank two, therefore, there exists an eigenvector corresponding to a zero mode which will be identified as the other involutive Hamiltonian. It is possible to show that this Hamiltonian density result of a linear combination of ${H^i}_2$ and H_4

$$H_5 = -\frac{2\pi}{k}\partial_j {H^j}_2 + H_4 \tag{25}$$

and take the form

$$H_5 = \partial_j\Pi^j + \frac{1}{4}\frac{k}{\pi}\varepsilon^{0ij}\partial_i A_j = 0. \tag{26}$$

Therefore, the set of involutive HJPDE are:

$$H_1 = \Pi^0 = 0, \tag{27a}$$

$$H_5 = \partial_i\Pi^i + \frac{k}{4\pi}\varepsilon^{0ij}\partial_i A_j = 0, \tag{27b}$$

and the non-involutive ones are:

$$H^i{}_2 = -\Pi^i + \frac{k}{4\pi}\varepsilon^{0ij}A_j = 0, \tag{28}$$

from which it is possible to define

$$M_{ij}\left(x,y\right) \equiv \left\{H^i{}_2\left(x\right), H^j{}_2\left(y\right)\right\} = -\frac{k}{2\pi}\varepsilon^{0ij}\delta^2\left(\mathbf{x}-\mathbf{y}\right). \tag{29}$$

Now, the parameters $\tau_{i,2}$ related to $H^i{}_2$ are eliminated if we invert the matrix with elements (29) and build the Generalizated Brackets (GB). Lets consider that this inverse $M^{-1}_{ij}\left(x,y\right)$ exist and satisfy

$$\int d^2z M_{ik}\left(x,z\right)\left(M^{kj}\right)^{-1}\left(z,y\right) = \delta_i^j\delta^2\left(\mathbf{x}-\mathbf{y}\right). \tag{30}$$

Using (29) in the equation (30) we determine

$$\left(M^{ij}\right)^{-1}\left(x,y\right) = \frac{2\pi}{k}\varepsilon^{0ij}\delta^2\left(\mathbf{x}-\mathbf{y}\right). \tag{31}$$

With the matrix $\left(M^{ij}\right)^{-1}$, we define the GB

$$\left\{A\left(x\right), B\left(y\right)\right\}^* = \left\{A\left(x\right), B\left(y\right)\right\} - \int d^2u\int d^2v\left\{A\left(x\right), H^k{}_2\left(u\right)\right\} \tag{32}$$
$$\left(M^{kl}\right)^{-1}\left(u,v\right)\left\{H^l{}_2\left(v\right), B\left(y\right)\right\},$$

which satisfy the condition,

$$\left\{A\left(x\right), H^k{}_2\left(y\right)\right\}^* = 0. \tag{33}$$

Thus, we obtain the following fundamental GB

$$\left\{A^\mu(x), A^\nu(y)\right\}^* = \frac{2\pi}{k}\varepsilon^{0\mu\nu}\delta^2(\mathbf{x}-\mathbf{y}), \tag{34a}$$

$$\left\{A^0(x), \Pi^0(y)\right\}^* = \delta^2(\mathbf{x}-\mathbf{y}), \tag{34b}$$

$$\left\{A^i(x), \Pi^j(y)\right\}^* = \frac{1}{2}\delta^{ij}\delta^2(\mathbf{x}-\mathbf{y}), \tag{34c}$$

$$\left\{\Pi^\mu(x), \Pi^\nu(y)\right\}^* = \frac{k}{8\pi}\varepsilon^{0\mu\nu}\delta^2(\mathbf{x}-\mathbf{y}). \tag{34d}$$

The GB (34) are a result of the IC for the Hamiltonian density $H^i{}_2$ and as consequence the parameter τ_1 has been removed by the dynamics defined in terms of (34). From the GB it is possible to show that Hamiltonian densities H_1 and H_5 are still maintained involutives.

3.2. *Characteristic Equations*

We have obtained the complete set of HJPDE, i.e., H_0, $\mathcal{G}^\lambda \equiv H_1$ and $\mathcal{G}^\omega \equiv H_5$ and each one is related to the parameters t, λ and ω. The last parameter was introduced to expanding the phase space such that the fundamental differential is now given by

$$dF(x) = \int d^2y \Big[\{F(x), H_0(y)\}^* dx^0 + \left\{F(x), \mathcal{G}^\lambda(y)\right\}^* d\lambda(y) + \{F(x), \mathcal{G}^\omega(y)\}^* d\omega(y)\Big]. \tag{35}$$

From where we obtain the characteristic equations

$$dA_0 = d\lambda, \tag{36}$$
$$dA_i = \partial_i A_0 dt - \partial_i d\omega, \tag{37}$$
$$d\Pi^0 = \frac{k}{2\pi}\varepsilon^{0ij}\partial_i A_j dt, \tag{38}$$
$$d\Pi^i = \frac{k}{4\pi}\varepsilon^{0ik}\partial_k A_0 dt - \frac{k}{4\pi}\varepsilon^{0ik}\partial_k d\omega. \tag{39}$$

This set of equations establishes that the dynamic evolution of the system depends on the parameters t, λ and ω. The IC states that these parameters are independents. Now, if we only consider the evolution in the temporal direction, we obtain

$$\partial_0 A_0 = 0, \tag{40}$$
$$\partial_0 A_i = \partial_i A_0, \tag{41}$$
$$\partial_0 \Pi^0 = \frac{k}{2\pi}\varepsilon^{0ij}\partial_i A_j, \tag{42}$$
$$\partial_0 \Pi^i = \frac{k}{4\pi}\varepsilon^{0ik}\partial_k A_0. \tag{43}$$

The equation (40) states that A_0 is indeterminate like the parameters of the theory. The relation (41) corresponds tho the dynamical equation (11) for $\alpha = i$. Meanwhile the relations (42) and (43) correspond to the IC for H_1 and ${H^i}_2$. Thus, we have established the correlation between the Euler Lagrange equations and the characteristics equations of the Hamilton-Jacobi formalism.

3.3. *Gauge transformations*

The evolution along the parameters λ and ω are identified as variations denoted with δ_λ and δ_ω they are given by

$$\delta_\lambda A_\mu = \delta^0_\mu \delta\lambda, \tag{44a}$$
$$\delta_\omega A_\mu = \delta^i_\mu \partial_i \delta\omega, \tag{44b}$$
$$\delta_\omega \Pi^i = -\frac{k}{4\pi}\varepsilon^{0ik}\partial_k \delta\omega, \tag{44c}$$

here, we are still considering that the parameters are linearly independents. In order to consider local transformations we must consider that this parameters λ and ω are no longer independent between them, but now depend on the time variable. Thus, we write the variations (44) as

$$\delta A_\mu = \delta^0_\mu \delta\lambda + \delta^i_\mu \partial_i \delta\omega, \tag{45}$$

$$\delta\Pi^i = -\frac{k}{4\pi}\varepsilon^{0ik}\partial_k \delta\omega. \tag{46}$$

The set of local variations (45) and (46) is now generated by the linear combinations of the involutive Hamiltonians, i.e.,

$$G^{\mathrm{can}} \equiv \int d^2y \left[\mathcal{G}^\lambda \delta\lambda + \mathcal{G}^\omega \delta\omega\right], \tag{47}$$

we call to G^{can} the generator of canonical transformations since we have that,

$$\delta A_\mu = \{A_\mu, G^{\mathrm{can}}\}^* = \delta^0_\mu \delta\lambda + \delta^i_\mu \partial_i \delta\omega, \tag{48}$$

$$\delta\Pi^i = \{A_\mu, G^{\mathrm{can}}\}^* = -\frac{k}{4\pi}\varepsilon^{0ik}\partial_k \delta\omega. \tag{49}$$

We know that the Chern-Simons theory has a groups of transformations that leave its action functional quasi-invariant. The variation of the action functional for fixed point variations is

$$\delta\mathscr{L}_{\mathrm{CS}} = \delta A_\mu \left[\frac{k}{4\pi}\varepsilon^{\mu\beta\gamma}F_{\beta\gamma}\right] + \partial_\mu\left[\frac{k}{4\pi}\delta A_\alpha \varepsilon^{\alpha\mu\gamma}A_\gamma\right]. \tag{50}$$

The generator of gauge transformations G^g will be related to the generator G^{can} when we analyze the condition that the variation (48) and (49) must satisfy to leave the theory quasi-invariant. For this we replace the fixed point variations in (50) by (48) and (49), thus the action becomes invariant under these transformations if

$$\delta\mathscr{L}_{\mathrm{CS}} = \left(\delta^0_\mu \delta\lambda + \delta^i_\mu \partial_i \delta\omega\right)\left[\frac{k}{4\pi}\varepsilon^{\mu\beta\gamma}F_{\beta\gamma}\right] + \partial_\mu\left[\frac{k}{4\pi}\delta A_\alpha \varepsilon^{\alpha\mu\gamma}A_\gamma\right] = 0. \tag{51}$$

Apart of a divergence, we have

$$\delta\mathscr{L}_{\mathrm{CS}} = \frac{k}{4\pi}\left(\varepsilon^{0ij}F_{ij}\delta\lambda + 2\varepsilon^{0ij}\partial_i F_{0j}\delta\omega\right). \tag{52}$$

From the Bianchi identity, we obtain that

$$\delta\mathscr{L}_{\mathrm{CS}} = \frac{k}{4\pi}\varepsilon^{0ij}F_{ij}\left(\delta\lambda - \partial_0 \delta\omega\right). \tag{53}$$

If the theory is invariant, i.e., $\delta\mathscr{L}_{\mathrm{CS}} = 0$, we should consider

$$\delta\lambda = \partial_0 \delta\omega, \tag{54}$$

hence, we have one arbitrary parameter, say $\Lambda \equiv \delta\omega$. Under the condition (54), the transformation on the fields A_μ takes the form

$$\delta A_\mu = \delta^0_\mu \partial_0 \Lambda + \delta^i_\mu \partial_i \Lambda = \partial_\mu \Lambda,$$

which is the well know invariance of the theory. From this condition we write the generator of the gauge transformation as

$$
\begin{aligned}
G^{can} \equiv \int d^2y \mathcal{G}^g &= \int d^2y \left(\mathcal{G}^\lambda \delta\lambda + \mathcal{G}^\omega \delta\omega\right) \\
&= \int d^2y \left(\mathcal{G}^\lambda \partial_0 + \mathcal{G}^\omega\right) \Lambda.
\end{aligned} \tag{55}
$$

We can check this affirmation by computing

$$
\{A_\mu, G^{can}\}^* = \delta A_\mu = \partial_\mu \Lambda. \tag{56}
$$

which is the well known gauge invariance of the Chern-Simons theory.

3.4. *Gauge Conditions*

At this stage, the non-involutive HJPDE have already been eliminated through the construction of the GB. Nevertheless, the dynamics is still not completely determined. As can be seen from the fundamental differential (35), the evolution of the system depends on the arbitrary parameters associated with the involutive Hamiltonians. These parameters are reflected in the characteristic equations (36)–(39), indicating that the evolution is not unique. This feature is the Hamilton-Jacobi counterpart of the arbitrariness that appears in the Dirac formalism when the dynamics is generated by the extended Hamiltonian.

Since the involutive HJPDE generate the canonical transformations of the theory, the remaining arbitrariness is directly associated with the gauge freedom of the system. When analyzing this temporal evolution through the relations (35), it becomes evident that the dynamics of the fields remain arbitrary precisely because the characteristic equations show an explicit dependence on the parameters λ and ω, which remain unfixed and independent of each other. According to Bergmann,[28] a physical observable must be interpreted as a function that is completely and uniquely determined by the dynamical equations and the initial conditions. Therefore, in order to recover predictability and obtain a unique dynamical evolution for the physical variables, it is necessary to impose additional conditions that fix the gauge.

The set of non-involutive constraints H_b, together with those involutive constraints $\mathcal{G}^a$ associated with the parameters entering the construction of the gauge transformation generator, define a reduced phase space. For most problems of physical relevance, isolating this space and explicitly identifying the coordinates that characterize it proves to be particularly challenging.

The physical observables are constructed in this space and, therefore, will depend on arbitrary functions. If we denote by $\bar{H}_r$ the set of all constraints associated with the system, that is: $\bar{H}_r = (\mathcal{G}^a, H_b)$, then an observable G is implicitly defined by the requirement that its Poisson bracket with all constraints vanishes,

$$
\{G, \bar{H}_r\} = g^s{}_r \bar{H}_s, \tag{57}
$$

In most cases, it is not possible to isolate the coordinates of the reduced phase space, and thus one must work with the gauge variables, namely those that are not observables. These variables do not commute under the Poisson bracket with all the constraints $\bar{H}_r$ and, consequently, their time evolution must be determined consistently by properly specifying the dependence on the parameters appearing in (16) or (35). The particular choice of these parameters, however, must have no influence whatsoever on the observables.

The existence of non-involutive constraints H_b allows to determine the value of its parameters in the fundamental differential (1) in terms of the time parameter that determine the evolution of any function of the phase space and that guarantees the construction of the GB (ref Non Involutive). The procedure to eliminate the gauge freedom inherent in the arbitrary parameters appearing in (35) consists in introducing additional constraints, namely the gauge conditions $\Omega^a(q,p) = 0$, which do not follow from the Lagrangian. In order to fix the values of the parameters, these gauge conditions must convert involutive constraints into non-involutive constraints, a requirement that is guaranteed if:

$$\det \left| \{ \Omega^a, \mathcal{G}^b \} \right| \neq 0. \tag{58}$$

This implies that the number of gauge-fixing conditions must equal the number of first-class constraints. Nevertheless, (58) constitutes only a necessary condition for the admissibility of the gauge constraints. In fact, many different gauge conditions may satisfy this requirement. Therefore, the gauge constraints must be further restricted to be admissible, they must be attainable in all and must fix the gauge uniquely.

Within the Hamilton-Jacobi framework, we implement the gauge-fixing procedure by enlarging the set of HJPDE. The gauge conditions are incorporated as additional Hamiltonians and are assigned independent parameters in the fundamental differential. The gauge conditions are incorporated as additional Hamiltonians and are assigned independent parameters in the fundamental differential. However, since there are initially no independent variables directly related to these gauge conditions, we must address this limitation by expanding the space of independent variables, thereby relating an arbitrary parameter to each new constraint. In this way, gauge fixing is not introduced as an external step after the constraint analysis, but becomes part of the integrability structure of the theory itself.

This procedure differs from previous implementations of gauge fixing within the Hamilton-Jacobi approach, where gauge conditions are introduced as additional constraints in order to construct GB and establish their equivalence with Dirac brackets.[27] In the present formulation, the gauge conditions are incorporated directly as additional Hamilton-Jacobi partial differential equations and associated with new independent parameters in the fundamental differential. This requires an enlargement of the parameter space and embeds the gauge-fixing procedure into the integrability structure of the theory itself.

Consequently, the enlarged set of Hamiltonians is no longer involutive. As a result of this breakdown of involutivity, all arbitrary parameters become completely determined through the integrability conditions, the characteristic equations define a unique evolution, and a new GB structure can be constructed. To illustrate this mechanism explicitly in a topological gauge theory, we propose to impose these conditions within the context of the pure $U(1)$ Chern-Simons theory,

$$A_0 = 0 \qquad , \qquad \partial_i A_i = 0. \tag{59}$$

These equations must be incorporated into the set of HJPDES defined by the fundamental differential (35), with each of them assigned an independent parameter. Consequently, the full set of constraints is:

$$\begin{aligned}
H_0 \equiv& p^0 - \frac{k}{2\pi}\varepsilon^{0ij} A_0 \left(\partial_i A_j\right) && \rightarrow \tau_0 = x^0, && (60)\\
H_1 \equiv& \Pi^0 && \rightarrow \tau_1, && (61)\\
H_2 \equiv& \partial_i \Pi^i + \frac{k}{4\pi}\varepsilon^{0ij}\partial_i A_j && \rightarrow \tau_2, && (62)\\
H_3 \equiv& A_0 && \rightarrow \tau_3, && (63)\\
H_4 \equiv& \partial_i A_i && \rightarrow \tau_4. && (64)
\end{aligned}$$

Then we define the dynamics of the system by the new fundamental differential

$$dF(x) = \int d^2y \left\{ F(x), H_\sigma(y) \right\}^* d\tau_\sigma(y), \tag{65}$$

with $\{\sigma\} = \{0, ..., 4\}$. Now, the set of HJPDE (H_1, H_2, H_3, H_4) is not in involution, since it is possible to construct the following matrix $\Phi_{ij}(\mathbf{x}, \mathbf{y}) \equiv \{H_i(x), H_j(y)\}^*$, which turns out to be non-singular, with its inverse given by

$$\Phi^{-1}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{k}{2\pi}\frac{1}{\nabla_x^2} \\ -1 & 0 & 0 & 0 \\ 0 & \frac{k}{2\pi}\frac{1}{\nabla_x^2} & 0 & 0 \end{pmatrix} \delta^2(\mathbf{x} - \mathbf{y}), \tag{66}$$

The regularity of the matrix implies that the independent variables $(\tau_1, \tau_2, \tau_3, \tau_4)$ can be expressed as functions of the parameter $\tau_0 = t$, namely

$$d\tau_i\left(y\right) = -\iint d^2u d^2v \Phi_{ij}^{-1}(y, v) \left\{H_j(v), H_0(u)\right\}^* dt. \tag{67}$$

Accordingly, the GB of the system can be defined as

$$\begin{aligned}
\left\{ F\left(x\right), G\left(y\right)\right\}^\bigstar \equiv \left\{ F\left(x\right), G\left(y\right)\right\}^* - \iint d^2u d^2v \left\{ F\left(x\right), H_i\left(u\right)\right\}^* \\
\Phi_{ij}^{-1}\left(u, v\right) \left\{ H_j\left(v\right), G\left(y\right)\right\}^*. \qquad (68)
\end{aligned}$$

The only non-vanishing fundamental GB are given by

$$\left\{ A_i\left(x\right), A_j\left(y\right)\right\}^\bigstar = \left[\frac{2\pi}{k}\varepsilon_{0ij} - \frac{2\pi}{k}\left(\varepsilon_{0mj}\partial_m^x\partial_i^x - \varepsilon_{0mi}\partial_m^x\partial_j^x\right)\right]\delta^2\left(\mathbf{x} - \mathbf{y}\right). \tag{69}$$

These expressions agree with the results obtained through Dirac's method (see Eq. (A.6)).

4. Hamilton-Jacobi Analysis for Maxwell-Chern-Simons Theory

The theory is described by adding the Chern-Simons term to the electromagnetic field, so that its Lagrangian density reads

$$\mathscr{L}_{\mathrm{MCS}} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{k}{8\pi}\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta}A_{\gamma}, \tag{70}$$

so that the Euler-Lagrange equations of motion can be expressed as

$$\frac{k}{4\pi}\varepsilon^{\alpha\beta\nu}F_{\beta\nu} + \partial_{\beta}F^{\beta\alpha} = 0. \tag{71}$$

The canonical momentas are given by

$$\Pi^{\alpha} = \frac{\partial\mathscr{L}_{\mathrm{MCS}}}{\partial\left(\partial_0 A_{\alpha}\right)} = -F^{0\alpha} + \frac{k}{4\pi}\varepsilon^{0\alpha\rho}A_{\rho}. \tag{72}$$

4.1. *Non-involutive HJPDE*

The antisymmetry of the Levi–Civita symbol implies that the temporal component of the canonical momentum is a canonical constraint: $\Pi^0 = 0$. Using (70) and (72), the canonical Hamiltonian can be constructed, and it takes the form

$$\begin{aligned}\mathscr{H}_{\mathrm{MCS}} =& \frac{1}{2}\left(\Pi^{i} - \frac{k}{4\pi}\varepsilon^{0ij}A_{j}\right)^{2} + \frac{1}{2}\left(\varepsilon^{0ij}\partial_{i}A_{j}\right)^{2} \\ &- \frac{k}{4\pi}\varepsilon^{0ij}A_{0}(\partial_{i}A_{j}) - A_{0}\partial_{i}\Pi^{i}.\end{aligned} \tag{73}$$

Accordingly, the theory is defined by the following set of HJPDE

$$H_0 \equiv p^0 + \mathscr{H}_{\mathrm{MCS}} = 0, \qquad \rightarrow \tau_0 = x^0, \tag{74a}$$

$$H_1 \equiv \Pi^0 = 0, \qquad \rightarrow \tau_1. \tag{74b}$$

For testing the integrability condition of (74), we start by defining the fundamental differential

$$dF(x) = \int d^2y\{F(x), H_{\sigma}(y)\}d\tau_{\sigma}(y), \tag{75}$$

where $\{\sigma = 0, 1\}$. The Hamiltonians H_1 are involutive among themselves, since $\{H_1(x), H_1(y)\} = 0$ and therefore their integrability condition depends only on the Poisson brackets with the Hamiltonian H_0. In this case, we obtain

$$dH_1 = \left[\frac{k}{4\pi}\varepsilon^{0kl}\partial_k A_l + \partial_k\Pi^k\right]dt = 0, \tag{76}$$

therefore, in order to ensure the consistency of the formalism, we define a new Hamiltonian

$$H_2 \equiv \partial_k\Pi^k + \frac{k}{4\pi}\varepsilon^{0kl}\partial_k A_l = 0. \tag{77}$$

which is also in involution. Once the integrability condition is fully satisfied, the system is characterized by a closed set of involutive Hamiltonians, namely H_0, H_1, and H_2.

4.2. *Characteristic equations and Gauge Transformations*

Since the Hamiltonian H_2 arises as a consequence of the integrability condition imposed on H_1, it is not canonical and does not have a correspondent variable in the original phase-space. In this case we need to expand our initial phase space in order to include this Hamiltonian. If we introduced ω as a new parameters which is related to the Hamiltonian H_2. If, If we perform a redefinition of the Hamiltonians $\mathcal{G}^\lambda \equiv H_1$ and $\mathcal{G}^\omega \equiv H_2$, then the fundamental differential, constructed from the complete set of involutive Hamiltonians, is given by

$$dF(x) = \int d^2y \Big[\{F(x), H_0(y)\} dt + \left\{F(x), \mathcal{G}^\lambda(y)\right\} d\lambda(y) + \{F(x), \mathcal{G}^\omega(y)\} d\omega(y) \,. \tag{78}$$

From this equation, one derives the following set of CEs,

$$dA_\mu = \left[\Pi^i - \frac{k}{4\pi}\varepsilon^{0im}A_m + \partial_i A_0\right] dt + \delta^i_\mu \partial_i d\omega + \delta^0_\mu d\lambda, \tag{79}$$

$$\begin{aligned} d\Pi^\mu = {} & \delta^\mu_i \left[\partial^x_k F^{ki} - \frac{k}{4\pi}\varepsilon_{0ik}\Pi^k - \left(\frac{k}{4\pi}\right)^2 A_i + \frac{k}{4\pi}\varepsilon^{0ik}\partial_k A_0\right] dt \\ & + \delta^\mu_0 \left[\frac{k}{4\pi}\varepsilon^{0kl}\partial_k A_l + \partial_k \Pi^k\right] dt - \delta^\mu_i \frac{k}{4\pi}\varepsilon^{0ik}\partial_k d\omega. \end{aligned} \tag{80}$$

These equations describe the dynamical evolution of the system depending on the parameters t, λ^a and ω^a. Frobenius' theorem Condition states that these parameters are independents, therefore the evolution in the direction of a given parameter is independent of the evolution along the others. Then, if we restrict our analysis to the evolution along the temporal direction, we obtain

$$\partial_0 A^a_0 = 0, \tag{81}$$

$$\partial_0 A^a_i = \Pi^i - \frac{k}{4\pi}\varepsilon^{0im}A_m + \partial_i A_0, \tag{82}$$

$$\partial_0 \pi^0_a = \partial_k \Pi^k + \frac{k}{4\pi}\varepsilon^{0kl}\partial_k A_l, \tag{83}$$

$$\partial_0 \pi^i_a = \partial_j F^{ji} - \frac{k}{4\pi}\varepsilon_{0ij}\Pi^j - \left(\frac{k}{4\pi}\right)^2 A_i + \frac{k}{4\pi}\varepsilon^{0ij}\partial_j A_0. \tag{84}$$

Now, the evolution along the parameters λ and ω represents canonical transformations and is given by

$$\delta_\lambda A_\mu = \delta^0_\mu \delta\lambda \tag{85a}$$

$$\delta_\omega A_\mu = \delta^i_\mu \partial_i \delta\omega, \tag{85b}$$

$$\delta_\omega \pi^i = -\frac{k}{4\pi}\delta^\mu_i \varepsilon^{0ik}\partial_k \delta\omega, \tag{85c}$$

In order to study local transformation, we must consider that the parameters λ^a and ω^a are no longer independent between them, but now depend on the time variable

x^0. We write the variations (85) as

$$\delta A_\mu = \delta^0_\mu \delta\lambda + \delta^i_\mu \partial_i \delta\omega, \tag{86a}$$

$$\delta_\omega \pi^i = -\frac{k}{4\pi}\delta^\mu_i \varepsilon^{0ik}\partial_k\delta\omega. \tag{86b}$$

We have used the symbol δ to denote the variations δ_λ and δ_ω, which are correlated. The local variations (86) are generated by a linear combination of the involutive Hamiltonians

$$G^{can} \equiv \int d^3y \left[\mathcal{G}^\lambda \delta\lambda + \mathcal{G}^\omega \delta\omega\right], \tag{87}$$

we call G^{can} the generator of canonical transformations since we have that

$$\delta A_\mu = \{A_\mu, G^{can}\} = \delta^0_\mu \delta\lambda^a + \delta^i_\mu \partial_i \delta\omega, \tag{88a}$$

$$\delta\pi^i = \left\{\pi^i, G^{can}\right\} = -\frac{k}{4\pi}\delta^\mu_i \varepsilon^{0ik}\partial_k\delta\omega. \tag{88b}$$

It is well known that the theory possesses a group of transformations that render its action functional quasi-invariant. The connection between the generator of these gauge transformations and the canonical generator G^{can} arises from analyzing the conditions under which the variations (88) keep the theory quasi-invariant. Then, the action (70) becomes invariant under these transformations if

$$\begin{aligned} \delta\mathscr{L}_{\text{MCS}} &= \left(\delta^0_\alpha \delta\lambda + \delta^j_\alpha \partial_j \delta\omega\right)\left(\partial_\nu F^{\nu\alpha} + \frac{k}{4\pi}\,\varepsilon^{\alpha\beta\gamma}F_{\beta\gamma}\right) \\ &\quad -\partial_\mu\left[\delta A_\alpha\left(F^{\mu\alpha} + \frac{k}{4\pi}\,\varepsilon^{\mu\alpha\gamma}A_\gamma\right)\right] \\ &= 0. \end{aligned} \tag{89}$$

Using the Bianchi identities we obtain

$$\delta\mathscr{L}_{\text{MCS}} = \left(\partial_j F^{j0} + \frac{k}{4\pi}\,\varepsilon^{0jk}F_{jk}\right)\left(\delta\lambda - \partial_0\delta\omega\right). \tag{90}$$

If the theory is invariant, we should consider

$$\delta\lambda = \partial_0\delta\omega, \tag{91}$$

therefore there is a unique independent parameter, say $\delta\omega = \Lambda$. Under the condition (91), the transformation on the fields takes the form

$$\delta A_\mu = \delta^0_\mu \partial_0\Lambda + \delta^i_\mu \partial_i \delta\Lambda = \partial_\mu\Lambda, \tag{92}$$

from which it can be determined that

$$\delta A_\mu = \{A_\mu, G^{can}\} = \partial_\mu\Lambda. \tag{93}$$

4.3. *Generalized Brackets*

The gauge-fixing procedure follows the same strategy developed for the pure Chern-Simons theory. The gauge conditions are incorporated into the Hamilton-Jacobi scheme as additional Hamiltonians, enlarging the set of HJPDE and converting the resulting system into a non-involutive one. This allows the arbitrary parameters associated with the gauge freedom to be completely determined and leads to a GB structure equivalent to the Dirac bracket formulation.

In the context of Maxwell-Chern-Simons theory, we propose to impose the following gauge-fixing conditions (see Eq. (A.7)),

$$\partial_i A_i = 0, \qquad \partial_i \Pi^i - \frac{k}{4\pi}\varepsilon^{0ij}\partial_i A_j + \partial_i\partial_i A_0 = 0, \tag{94}$$

These equations must be incorporated into the set of HJPDE defined by the fundamental differential, with each of them assigned an independent parameter. Consequently, the full set of constraints is:

$$\begin{aligned}
H_0 =& p^0 + \mathscr{H}^0_{MCS} && \rightarrow t, \\
H_1 =& \Pi^0 && \rightarrow \tau_1, \\
H_2 =& \frac{k}{4\pi}\varepsilon^{0kl}\partial_k A_l + \partial_k \Pi^k && \rightarrow \tau_2, \\
H_3 =& \partial_i\partial_i A_0 + \frac{k}{4\pi}\varepsilon^{0kl}\partial_k A_l + \partial_k \Pi^k && \rightarrow \tau_3, \\
H_4 =& \partial_i A_i && \rightarrow \tau_4.
\end{aligned} \tag{95}$$

Now, the dynamics is determined by the fundamental differential

$$dF(x) = \int d^2y \left\{ F(x), H_\sigma(y) \right\} d\tau_\sigma(y), \tag{96}$$

with $\{\sigma\} = \{0, ..., 4\}$. The set of HJPDE (H_1, H_2, H_3, H_4) is not in involution, since it is possible to construct the matrix $\Phi_{ij}(\mathbf{x}, \mathbf{y}) \equiv \{H_i(x), H_j(y)\}$, which turns out to be non-singular, with its inverse given by $\Phi(\mathbf{x}, \mathbf{y})$,

$$\Phi^{-1}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \frac{1}{\nabla_x^2}\delta^2(\mathbf{x} - \mathbf{y}), \tag{97}$$

The regularity of the matrix implies that the independent variables $(\tau_1, \tau_2, \tau_3, \tau_4)$ can be expressed as functions of the parameter $\tau_0 = t$. Now, the GB of the system can be defined as

$$\begin{aligned}
\left\{ F(x), G(y) \right\}^{\bigstar} \equiv& \left\{ F(x), G(y) \right\} - \iint d^2u d^2v \left\{ F(x), H_i(u) \right\} \\
& \Phi^{-1}_{ij}(u, v) \left\{ H_j(v), G(y) \right\}.
\end{aligned} \tag{98}$$

The non-vanishing fundamental GB are given by

$$\{A_i(x), \Pi_j(y)\}^\star = \left(\delta_{ij} - \partial_i^x \partial_j^x \frac{1}{\nabla_x^2}\right) \delta^2(\mathbf{x} - \mathbf{y}), \tag{99}$$
$$\{\Pi_i(x), \Pi_j(y)\}^\star = \frac{k}{4\pi}\left(\varepsilon_{0mi}\partial_m^x \partial_j^x - \varepsilon_{0mj}\partial_m^x \partial_i^x\right) \frac{1}{\nabla_x^2}\delta^2(\mathbf{x} - \mathbf{y}).$$

These expressions agree with the results obtained through Dirac's method (see Eq. (A.8)).

An important feature of the present formulation is the manner in which gauge-fixing conditions are incorporated into the Hamilton-Jacobi framework. Rather than introducing them only as auxiliary constraints for the construction of generalized brackets, they are promoted to additional Hamilton-Jacobi equations and included directly in the fundamental differential. This enlarges the parameter space of the theory and allows gauge fixing to be interpreted as part of the integrability analysis itself. The resulting GB are fully consistent and reproduce the same physical content obtained through the Dirac formalism.

Conclusions

In this work, we analyzed the Hamiltonian structure of the U(1) Chern-Simons and Maxwell-Chern-Simons theories in $(2+1)$ dimensions within the Hamilton-Jacobi formalism. For both models, we identified the complete set of Hamilton-Jacobi partial differential equations, determined the involutive and non-involutive sectors of the theory, constructed the corresponding generalized brackets, and obtained the generators of canonical and gauge transformations. The resulting generalized brackets and gauge transformations were shown to reproduce the structures obtained within the Dirac formalism.

A central result of this work is the incorporation of gauge-fixing conditions into the integrability structure of the Hamilton-Jacobi formalism. After eliminating the original non-involutive Hamiltonians through generalized brackets, the fundamental differential still depends on arbitrary parameters associated with the involutive Hamiltonians, reflecting the gauge freedom of the theory. By promoting the gauge-fixing conditions to additional Hamilton-Jacobi equations and introducing independent parameters associated with them, the enlarged system becomes non-involutive and all arbitrary parameters are determined through the integrability conditions. This construction naturally leads to a generalized bracket structure equivalent to the corresponding Dirac brackets and establishes a direct relation between gauge fixing and integrability within the Hamilton-Jacobi framework.

Beyond reproducing the results obtained in the Dirac formalism, our analysis shows that the Hamilton-Jacobi approach provides a consistent and unified framework in which the constraint structure, integrability conditions, characteristic equations, and generators of gauge transformations emerge from a common geometric setting. In this perspective, gauge fixing is incorporated as part of the integrability structure rather than introduced as an external procedure after the constraint analysis. In particular, the topological nature of the Chern-Simons theory and the emergence of a topological mass in the Maxwell-Chern-Simons model are naturally exhibited.

Although the present work focuses on the Chern-Simons and Maxwell-Chern-Simons theories, the construction is formulated at the level of the Hamilton-Jacobi integrability structure and is therefore not intrinsically restricted to these examples. Future applications to other constrained gauge systems may provide further insight into the scope and usefulness of this approach.

Appendix A. Dirac Formulation

The constrained Hamiltonian analysis is performed following Dirac's systematic procedure for systems with constraints. Let $\phi_\alpha(x) \approx 0$ denote the complete set of constraints of the theory. When these constraints form a second-class system, one defines the constraint matrix

$$C_{\alpha\beta}(x,y) = \{\phi_\alpha(x), \phi_\beta(y)\}, \tag{A.1}$$

which, provided it is non-singular, admits a functional inverse. The Dirac bracket between two arbitrary functionals F and G is then defined as

$$\begin{aligned}\{F,G\}_D = \{F,G\} - \int d^2u\, d^2v\, \{F, \phi_\alpha(u)\}\\ (C^{-1})^{\alpha\beta}(u,v)\{\phi_\beta(v), G\},\end{aligned} \tag{A.2}$$

thereby ensuring that all second-class constraints hold strongly.

Pure Chern-Simons theory

For the pure Chern-Simons theory, the Hamiltonian analysis yields the first-class constraints

$$\Pi^0 \approx 0, \qquad \partial_i\Pi_i - \frac{k}{4\pi}\varepsilon_{0ij}\partial_i A_j \approx 0, \tag{A.3}$$

together with the second-class constraints

$$\Pi_i + \frac{k}{4\pi}\varepsilon_{0ij}A_j \approx 0. \tag{A.4}$$

To remove the gauge degrees of freedom associated with the first-class constraints, we impose the gauge-fixing conditions

$$A_0 \approx 0, \qquad \partial_i A_i \approx 0. \tag{A.5}$$

The full set of constraints thus becomes second class, and the associated matrix $C_{\alpha\beta}$ is invertible. The Dirac bracket can therefore be constructed explicitly from Eq. (A.2). One obtains

$$\begin{aligned}\{A_i(x), A_j(y)\}_D = \Bigg[\frac{2\pi}{k}\varepsilon_{0ij} - \frac{2\pi}{k}\left(\varepsilon_{0mj}\partial^x_m\partial^x_i - \varepsilon_{0mi}\partial^x_m\partial^x_j\right)\\ \frac{1}{\nabla^2_x}\Bigg]\,\delta^2(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{A.6}$$

Maxwell-Chern-Simons theory

In the Maxwell-Chern-Simons theory, the set of first-class constraints coincides with that of the pure Chern-Simons theory, while primary second-class constraints are absent at the outset. To implement the Dirac procedure, we impose the gauge conditions

$$\partial_i A_i \approx 0, \qquad \partial_i \Pi^i - \frac{k}{4\pi}\varepsilon^{0ij}\partial_i A_j + \partial_i \partial_i A_0 \approx 0, \tag{A.7}$$

where the second condition follows from requiring preservation in time of the first within Dirac's consistency algorithm. The resulting enlarged set of constraints is again second class, with non-singular matrix $C_{\alpha\beta}$. The fundamental non-vanishing Dirac brackets are

$$\begin{aligned} \{A_i(x), \Pi_j(y)\}_D &= \left(\delta_{ij} - \partial_i^x \partial_j^x \frac{1}{\nabla_x^2}\right) \delta^2(\mathbf{x}-\mathbf{y}), \\ \{\Pi_i(x), \Pi_j(y)\}_D &= \frac{k}{4\pi}\left(\varepsilon_{0mi}\partial_m^x \partial_j^x - \varepsilon_{0mj}\partial_m^x \partial_i^x\right) \\ &\quad \frac{1}{\nabla_x^2}\delta^2(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{A.8}$$

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