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Topologically massive Yang-Mills: A Hamilton-Jacobi constraint analysis

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We analyse the constraint structure of the topologically massive Yang-Mills theory in instant-form and null-plane dynamics via the Hamilton-Jacobi formalism. The complete set of hamiltonians that generates the dynamics of the system is obtained from the Frobenius' integrability conditions, as well as its characteristic equations. As generators of canonical transformations, the hamiltonians are naturally linked to the generator of Lagrangian gauge transformations. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4870641]

I. INTRODUCTION

It is well known that $2 + 1$ dimensional non-Abelian gauge theories suffer from infrared divergences. One way to avoid these divergences is by adding a Chern-Simons (CS) term to their actions, building topologically massive theories.[1,2] As examples of such models we have the Maxwell-Chern-Simons (MCS) theory, the topologically massive gravity (TMG), and the topologically massive Yang-Mills (TMYM) theory. As result of adding the CS term it is provided mass for the fields while retaining their gauge invariance. At the quantum level the topological mass provides an infrared cut-off, getting rid of the infrared divergences. They are also useful in models of condensed matter, e.g., the quantum Hall effect,[3] superconductivity,[4] and four dimensional high temperature gauge theories.[5] It is also important to highlight that there is a correspondence between these theories and the Self-Dual model.[6]

As gauge theories, topologically massive models are singular systems, so the study of their hamiltonian dynamics requires methods of constraint analysis. The study of constrained systems began with the works of Dirac[7] and Bergmann,[8] and resulted in the so called Dirac's method of constraint analysis,[9] which has become a powerful tool to deal with gauge theories. Regarding topological theories, the TMYM was studied using Dirac's method in Ref. 10. The MCS theory has been quantized using the Dirac bracket quantization, and also the Schwinger action principle in Ref. 11. More recently, the MCS and the TMYM theories have been analysed using the first-order formalism.[12]

Alternatives to Dirac's method have been developed over the years, e.g., the Faddev-Jackiw approach.[13] A more recent one is the Hamilton-Jacobi (HJ) formalism, first developed by Güler[14] and based on the Carathéodory's complete figure of the calculus of variations.[15] The main advantage

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of the HJ formalism is the fact that it consists in a complete formalism by itself, rather than a consistency method. Constrained systems are naturally described by Carathéodory's complete figure: the necessary and sufficient conditions for the existence of a stationary configuration of an action is reduced to a complete set of Hamilton-Jacobi partial differential equations, which defines a set of hamiltonians responsible for the dynamical evolution of a singular system. The completeness of this set is assured by the Frobenius' theorem, which implies that the hamiltonians must obey a set of integrability conditions. To better understand this formalism, several improvements[16–20] and applications[21–26] have been made. One of the desired features of the HJ formalism is the fact that no analogue of Dirac's conjecture is needed.

In this work, we intend to analyse the topologically massive Yang-Mills theory in instant-form and null-plane dynamics. The main objective is to find a complete set of involutive hamiltonians, to build its characteristic equations, and to find the generator of gauge transformations of the system in both forms of dynamics. The paper is organized as follows. In the Sec. II, we briefly present the HJ formalism. In Sec. III, we apply this procedure to the TMYM theory in the instant-form dynamics. In Sec. IV, we proceed similarly to the case of the null-plane dynamics. Finally, in Sec. V we discuss the results.

II. THE HAMILTON-JACOBI FORMALISM

Let us consider a system described by a Lagrangian function $L = L(x^i, \dot{x}^i, t)$, where $i = 1, 2, ...N$, which has a singular Hessian matrix of rank P. The singularity of the Hessian matrix separates the variables $x^i = (x^a, x^z)$, where $a = 1, 2, ..., P$ and $z = 1, 2, ..., R$ with $P + R = N$. The variables x^a are related to the invertible part of the Hessian, i.e., the definition of the canonical momenta $p_i \equiv \partial L/\partial \dot{x}^i$ allows expressions such as $\dot{x}^a = \dot{x}^a(x^a, x^z, p_a, p_z, t)$. The variables x^z, which are renamed to t^z for convenience, are related to its singular part, i.e., the definition of the conjugate momenta allows no expressions of the velocities $\dot{x}^z$ in terms of the other variables.

Following the Carathéodory's variational approach,[15] the necessary condition for extremizing the action $A = \int L dt$ is given by the existence of a function $S(x^a, x^z, t)$, solution of the equation

$$p_0 + p_a \dot{x}^a + p_z \dot{t}^z - L = 0, \qquad p_a \equiv \frac{\partial S}{\partial x^a}, \qquad p_z \equiv \frac{\partial S}{\partial t^z}, \qquad p_0 \equiv \frac{\partial S}{\partial t}. \tag{1}$$

We may define the canonical hamiltonian function by $H_0 = p_a \dot{x}^a + p_z \dot{t}^z - L$, which is explicitly independent of $\dot{t}^z$. In this case, Eq. (1) becomes the well known Hamilton-Jacobi equation $p_0 + H_0 = 0$, which is a first-order partial differential equation (PDE). On the other hand the singularity of the Hessian matrix assures that there are R canonical constraints $p_z + H_z = 0$, where $H_z \equiv -\partial L/\partial \dot{t}^z$. In the HJ formalism, these constraints form a set of R first-order PDEs. Renaming the variables as $t^\alpha \equiv (t^z, t = t^0)$, we are allowed to write all the PDEs in a unified way:

$$H'_\alpha\left(x^a, p_a, t^\alpha, p_\alpha\right) \equiv p_\alpha + H_\alpha\left(x^a, p_a, t^\alpha\right) = 0, \qquad \alpha = 0, 1, 2, ..R. \tag{2}$$

The functions H'_α form a set of $R + 1$ hamiltonian functions, and the set $H'_\alpha = 0$ are called the Hamilton-Jacobi partial differential equations (HJPDEs) of the system.

As first-order PDEs, and considering t^α as independent variables, the HJ equations (2) have a set of first-order total differential equations related to them. They are denoted as the characteristic equations (CEs)

$$dx^a = \frac{\partial H'_\alpha}{\partial p_a} dt^\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial x^a} dt^\alpha, \quad dS = p_a dx^a - H_\alpha dt^\alpha. \tag{3}$$

Solutions of the first pair of equations represent curves that are functions of $R + 1$ parameters t^α on the reduced phase space defined by the conjugated variables (x^a, p_a). From (3), and considering t^α as independent parameters, the evolution of any phase space function $F = F(x^a, t^\alpha, p_a, p_\alpha)$ is given by the fundamental differential

$$dF = \{F, H'_\alpha\} dt^\alpha, \tag{4}$$

where the Poisson brackets (PB) are defined on the extended phase space of the variables $(x^a, t^\alpha, p_a, p_\alpha)$:

$$\{A, B\} \equiv \frac{\partial A}{\partial t^\alpha}\frac{\partial B}{\partial p_\alpha} - \frac{\partial B}{\partial t^\alpha}\frac{\partial A}{\partial p_\alpha} + \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_a} - \frac{\partial B}{\partial x^a}\frac{\partial A}{\partial p_a}. \tag{5}$$

Therefore, from the CEs we notice that the H'_α play the role of generators of the dynamical evolution of the system. This is why we call them hamiltonians. For a regular system, the first pair of CE (3) becomes Hamilton's equations.

The existence of a complete solution of the set of HJPDEs, which also implies on independence between the parameters t^α, is guaranteed by the Frobenius' theorem:[20] a set of hamiltonians H'_α form a complete set of HJ equations $H'_\alpha = 0$ iff

$$\{H'_\alpha, H'_\beta\} = C_{\alpha\beta}^{\ \ \gamma} H'_\gamma \quad \Longleftrightarrow \quad dH'_\alpha = 0, \tag{6}$$

i.e., the hamiltonians must close a Lie algebra with the PB. The conditions (6) are called integrability conditions (ICs), and any set of functions satisfying these conditions are called involutive.

The presence of hamiltonians that does not satisfy the ICs, denoted as non-involutive hamiltonians, implies that the system of HJPDEs is not complete, or that the parameters t^α are not linearly independent. In this case the ICs may provide new HJ equations to complete the system, and may also indicate the dependence between the parameters. Dependent parameters are eliminated with the method outlined in Ref. 20, by analysing the singularity of the matrix with elements $M_{xy} \equiv \{H'_x, H'_y\}$. If this matrix is singular of rank $K \leq R$, there is a regular sub-matrix $M_{\bar{a}\bar{b}}$, with $\bar{a} = 1, 2, \ldots K$, and we may define the generalized brackets (GB)

$$\{A, B\}^* = \{A, B\} - \{A, H'_{\bar{a}}\} M^{-1}_{\bar{a}\bar{b}} \{H'_{\bar{b}}, B\}. \tag{7}$$

Then we may rewrite the fundamental differential as

$$dF = \{F, H'_{\bar{\alpha}}\}^* dt^{\bar{\alpha}}, \qquad \bar{\alpha} = 0, K+1, \ldots, R. \tag{8}$$

After the procedure of finding possible new hamiltonians and eliminate the dependence between the parameters, the hamiltonians $H'_{\bar{\alpha}}$ become the complete set of involutive hamiltonians of the system, this time with the PB substituted by the GB.

III. THE HJ ANALYSIS IN THE INSTANT-FORM

The addition of a Chern-Simons term to a three dimensional Yang-Mills action results in the so called topologically massive Yang-Mills theory:

$$S = \int d^3x \left[-\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \frac{\mu}{4} \varepsilon^{\mu\nu\gamma} \left(F_{a\mu\nu} A^a_\gamma - \frac{g}{3} f_{abc} A^a_\mu A^b_\nu A^c_\gamma \right) \right], \tag{9}$$

where μ is the mass parameter and the components of the field strength are defined by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{\ bc} A^b_\mu A^c_\nu. \tag{10}$$

Let us define the covariant derivative

$$D^{ab}_\mu \equiv \delta^{ab}\partial_\mu - g f^{ab}_{\ \ c} A^c_\mu \tag{11}$$

and the notation $D^{ab}_\mu X_b \equiv \left[D_\mu X\right]^a$. In this case, the field equations are given by

$$\left[D_\nu F^{\mu\nu}\right]_a - \frac{\mu}{2} \varepsilon^\mu_{\ \nu\gamma} F_a^{\nu\gamma} = 0. \tag{12}$$

From (10) and (11) we also obtain the Bianchi identities

$$\left[D_\alpha F_{\beta\mu}\right]^a + \left[D_\mu F_{\alpha\beta}\right]^a + \left[D_\beta F_{\mu\alpha}\right]^a = 0. \tag{13}$$

In the instant-form dynamics, the evolution of a relativistic field theory is given by the time evolution of field configurations in Cauchy surfaces defined by constant time surfaces $\tau \equiv x^0$. Here, x^0 means the zeroth component of a coordinate system $x^\mu = (x^0, x^1, x^2)$ in 2 + 1 dimensions. The metric used here is the one of signature $(+, -, -)$. In this form of dynamics the conjugate momenta of the action (9) are written by

$$\pi_a^{\alpha} = -F^{a0\alpha} + \frac{\mu}{2}\varepsilon^{0\alpha\gamma}A_\gamma^a.$$

The temporal component of the canonical momenta is a canonical constraint, $\pi_a^0 = 0$, just as the free Yang-Mills field. For the spatial components, a dynamical relation is obtained

$$\pi_a^i = -\eta^{ij}F_{0j}^a + \frac{\mu}{2}\varepsilon^{ij}A_j^a, \quad \varepsilon^{ij} \equiv \varepsilon^{0ij}, \tag{14}$$

where $i = 1, 2$. Equation (14) can be inverted to obtain

$$\partial_0 A_i^a = -\eta_{ij}\pi^{aj} + [D_i A_0]^a + \frac{\mu}{2}\eta_{ij}\varepsilon^{jk}A_k^a. \tag{15}$$

Therefore, the variables A_i^a are the ones related to the invertible part of the Hessian matrix of this system, while A_0^a are parameters of the theory.

The canonical hamiltonian density has the form

$$\mathcal{H}_0 = -\frac{1}{2}\eta_{ij}\left[\pi_a^i - \frac{\mu}{2}\varepsilon^{ik}A_{ak}\right]\left[\pi^{aj} - \frac{\mu}{2}\varepsilon^{jl}A_l^a\right] + \frac{1}{4}F_{ij}^a F_a^{ij} - A_0^a\left\{\left[D_i\pi^i\right]_a + \frac{\mu}{2}\varepsilon^{ij}\partial_i A_{aj}\right\}. \tag{16}$$

Therefore, the TMYM theory is characterised by the following set of HJPDEs:

$$\mathcal{H}' \equiv \pi + \mathcal{H}_0 = 0 \quad \rightarrow \quad x^0, \tag{17a}$$

$$\mathcal{H}_a'^0 \equiv \pi_a^0 = 0 \quad \rightarrow \quad A_0^a \equiv \lambda^a. \tag{17b}$$

Here, we have renamed $A_0^a \equiv \lambda^a$ since these fields now act as independent parameters. Moreover, we have $\pi \equiv \partial S/\partial x^0$ and $\pi_a^0 \equiv \partial S/\partial A_0^a$. The hamiltonian densities $\mathcal{H}'$ and $\mathcal{H}_a'^0$ are related to the parameters x^0 and λ^a, respectively. In order to test the ICs for (17) we define the fundamental Poisson Brackets $\left\{A_\mu^a(x), \pi_b^\nu(y)\right\} = \delta_b^a\delta_\mu^\nu\delta(x-y)$. With this structure, we build the fundamental differential

$$dF(x) = \int d^3y\left[\left\{F(x), \mathcal{H}'(y)\right\}dx^0 + \left\{F(x), \mathcal{H}_a'^0(y)\right\}d\lambda^a\right].$$

The hamiltonians $\mathcal{H}_a'^0$ are involutive among themselves, since $\left\{\mathcal{H}_a'^0(x), \mathcal{H}_b'^0(y)\right\} = 0$. Their integrability depends on the PB with the hamiltonian $\mathcal{H}'$. In this case, we have

$$d\mathcal{H}_a'^0(x) = \int d^3y\left\{\mathcal{H}_a'^0(x), \mathcal{H}'(y)\right\}dx^0 = \left(\left[D_i\pi^i\right]_a + \frac{\mu}{2}\varepsilon^{ij}\partial_i A_{aj}\right)dx^0 = 0,$$

therefore, we define a new hamiltonian

$$\mathcal{C}_a' \equiv \left[D_i\pi^i\right]_a + \frac{\mu}{2}\varepsilon^{ij}\partial_i A_{aj} = 0. \tag{18}$$

$\mathcal{C}_a'$ are also in involution, since they satisfy the Lie algebra

$$\left\{\mathcal{C}_a'(x), \mathcal{C}_b'(y)\right\} = f_{ab}{}^c\mathcal{C}_c'(x)\delta(x-y), \tag{19}$$

and their other PBs are zero. Therefore, we end up with the following set of involutive hamiltonians

$$\mathcal{H}' \equiv \pi + \mathcal{H}_0 = 0, \tag{20a}$$

$$\mathcal{H}_a'^0 \equiv \pi_a^0 = 0, \tag{20b}$$

$$\mathcal{C}_a' \equiv \left[D_i\pi^i\right]_a + \frac{\mu}{2}\varepsilon^{ij}\partial_i A_{aj}. \tag{20c}$$

A. The characteristic equations

Since $\mathcal{C}'_a$ results from the integrability condition of $\mathcal{H}'^0_a$, it is not canonical and does not have a correspondent variable in the original phase-space. In this case we may expand the parameter space with a new set of parameters ω^a, which will be related to the functions $\mathcal{C}'_a$. Let us rename the hamiltonians $\mathcal{G}^\lambda_a \equiv \mathcal{H}'^0_a$ and $\mathcal{G}^\omega_a \equiv \mathcal{C}'_a$. Then the fundamental differential, which involves all the involutive hamiltonians, has the following form

$$dF\,(x) = \int d^3y \left[\left\{F(x), \mathcal{H}'(y)\right\} dx^0 + \left\{F(x), \mathcal{G}^\lambda_a(y)\right\} d\lambda^a + \left\{F(x), \mathcal{G}^\omega_a(y)\right\} d\omega^a\right]. \tag{21}$$

From this equation we obtain the following set of CEs

$$dA^a_\mu = \delta^k_\mu\left[-\eta_{jk}\pi^{aj} + [D_kA_0]^a + \frac{\mu}{2}\eta_{ij}\varepsilon^{jk}A^a_k\right]dx^0 + \delta^0_\mu d\lambda^a - \delta^k_\mu\,[D_kd\omega]^a\,, \tag{22a}$$

$$\begin{aligned} d\pi^\mu_a = {} & \delta^\mu_k\left(\left[D_jF^{jk}\right]_a + gf_a{}^{bc}\pi^k_bA_{c0} - \frac{\mu}{2}\varepsilon^{jk}\left[D_0A_j\right]_a\right)dx^0 \\ & + \delta^\mu_0\left(\left[D_k\pi^k\right]_a + \frac{\mu}{2}\varepsilon^{kj}\partial_kA_{aj}\right)dx^0 + \delta^\mu_k\left[gf_{abc}\pi^{ck} + \frac{\mu}{2}\delta_{ab}\varepsilon^{jk}\partial_j\right]d\omega^b. \end{aligned} \tag{22b}$$

These equations describe the dynamical evolution of the system depending on the parameters x^0, λ^a, and ω^a. Frobenius' theorem implies that these parameters are independent, therefore the evolution in the direction of a given parameter is independent of the evolution along the others. If we consider the evolution in the temporal direction we obtain

$$\partial_0A^a_0 = 0, \tag{23a}$$

$$\partial_0A^a_i = -\eta_{ij}\pi^{aj} + [D_iA_0]^a + \frac{\mu}{2}\eta_{ij}\varepsilon^{jk}A^a_k, \tag{23b}$$

$$\partial_0\pi^0_a = \left[D_i\pi^i\right]_a + \frac{\mu}{2}\varepsilon^{ij}\partial_iA_{aj}, \tag{23c}$$

$$\partial_0\pi^i_a = \left[D_jF^{ji}\right]_a + gf_{abc}\pi^{bi}A^c_0 - \frac{\mu}{2}\varepsilon^{ji}\left[D_0A_j\right]_a. \tag{23d}$$

Equation (23a) means that A^a_0 are constants, while Eq. (23b) is exactly equal to (15). Since the canonical momenta π^0_a are zero, we have that Eq. (23c) represents the ICs for $\mathcal{G}^\lambda_a \equiv \mathcal{H}'^0_a$. We notice that this equation is equivalent to the $\mu = 0$ component of (12), while (23d) corresponds to the field equations for $\mu = i$. Thus, we have established the equivalence between the field equations and the CEs for the TMYM theory.

B. Generator of gauge transformations

The temporal evolution of the characteristic equations is then equivalent to the field equations. The evolution along the parameters λ^a and ω^a represents canonical transformations, denoted with δ_λ and δ_ω, and is given by

$$\delta_\lambda A^a_\mu = \delta^0_\mu\delta\lambda^a, \tag{24a}$$

$$\delta_\omega A^a_\mu = -\delta^k_\mu\,[D_k\delta\omega]^a\,, \tag{24b}$$

$$\delta_\omega\pi^i_a = \left(gf_{abc}\pi^{ci} + \frac{\mu}{2}\varepsilon^{ji}\delta_{ab}\partial_j\right)\delta\omega^b. \tag{24c}$$

Since the parameters are time-independent, these variations are global canonical transformations. In order to study local transformations we must consider that the parameters λ^a and ω^a are no longer independent among themselves, and now also depend on the time variable x^0. In this case we rewrite the transformations (24) as

$$\delta A_\mu^a = \delta_\mu^0 \delta\lambda^a - \delta_\mu^k \left[D_k \delta\omega\right]^a , \tag{25a}$$

$$\delta\pi_a^i = g f_{abc} \pi^{ci} \delta\omega^b - \frac{\mu}{2} \varepsilon^{ij} \partial_j \delta\omega_a . \tag{25b}$$

We have now simply used the symbol δ since the variations δ_λ and δ_ω are correlated. The set of local variations (25) is now generated by the linear combination of the involutive hamiltonians.[27]

$$G^{can} \equiv \int d^3y \left[\mathcal{G}_a^\lambda \delta\lambda^a + \mathcal{G}_a^\omega \delta\omega^a\right]. \tag{26}$$

G^{can} is the generator of the canonical transformations (25), since

$$\delta A_\mu^a = \left\{A_\mu^a, G^{can}\right\} = \delta_\mu^0 \delta\lambda^a - \delta_\mu^k \left[D_k \delta\omega\right]^a , \tag{27a}$$

$$\delta\pi_a^i = \left\{\pi_a^i, G^{can}\right\} = g f_{abc} \pi^{ci} \delta\omega^b - \frac{\mu}{2} \varepsilon^{ij} \partial_j \delta\omega_a . \tag{27b}$$

It is well known that the TMYM theory has a group of transformations that leaves its action functional invariant. The generator G^g of these gauge transformations is in fact related to the generator G^{can}. Let us take the set of canonical transformations (27). The action (9) becomes invariant under these transformations if

$$\delta\mathcal{L} = \left(\delta_\alpha^0 \delta\lambda^a - \delta_\alpha^j \left[D_j\delta\omega\right]^a\right) \left(\left[D_\nu F^{\nu\alpha}\right]_a + \frac{\mu}{2}\varepsilon^{\alpha\beta\gamma} F_{a\beta\gamma}\right) - \partial_\mu \left[\delta A_\alpha^a \left(F_a^{\mu\alpha} + \frac{\mu}{2}\varepsilon^{\mu\alpha\gamma} A_{a\gamma}\right)\right] = 0.$$

Using the identity $X^a[D_\mu Y]_a = -Y^a[D_\mu X]_a$, apart of a divergence, we have

$$\begin{aligned}\delta\mathcal{L} &= \delta\lambda^a \left(\left[D_i F^{i0}\right]_a + \frac{\mu}{2}\varepsilon^{ij} F_{aij}\right) - \left[D_i\delta\omega\right]^a \left(\left[D_0 F^{0i}\right]_a + \left[D_j F^{ji}\right]_a - \mu\varepsilon^{ij} F_{a0j}\right), \\ &= F^{a0i}\left(\left[D_i\delta\lambda\right]_a + \left[D_0 D_i \delta\omega\right]_a\right) + F^{aji}\left[D_j D_i \delta\omega\right]_a + \mu\varepsilon^{ij}\left(F_{0j}^a \left[D_i\delta\omega\right]_a + \frac{1}{2}\delta\lambda^a F_{aij}\right).\end{aligned}$$

We also have $F^{a0i}[D_0 D_i \delta\omega]^a = F^{a0i}[D_i D_0 \delta\omega]_a$ and $F^{aji}[D_j D_i \delta\omega]_a = 0$. Then,

$$\delta\mathcal{L} = F_a^{0i} D_i^{ab}\left(\left[D_0\delta\omega\right]_b + \delta\lambda_b\right) + \mu\varepsilon^{ij}\left[-\delta\omega^a \left[D_i F_{0j}\right]_a + \frac{1}{2}\delta\lambda^a F_{aij}\right].$$

From the Bianchi identities (13) we obtain

$$\varepsilon^{ij}\left[D_0 F_{ij}\right]^a = -\varepsilon^{ij}\left(\left[D_j F_{0i}\right]^a + \left[D_i F_{j0}\right]^a\right) = 2\varepsilon^{ij}\left[D_i F_{0j}\right]^a .$$

This yields

$$\delta\mathcal{L} = \left(F_a^{0i} D_i^{ab} + \frac{\mu}{2}\varepsilon^{ij} F_{ij}^b\right)\left(\left[D_0\delta\omega\right]_b + \delta\lambda_b\right).$$

If the theory is invariant, i.e., $\delta\mathcal{L} = 0$, we should consider

$$\delta\lambda^a = -\left[D_0\delta\omega\right]^a , \tag{28}$$

therefore there is a unique independent parameter, say $\delta\omega^a = \Lambda^a$. Under the condition (28), the transformation on the fields A_μ^a takes the form

$$\delta A_\mu^a = -\delta_\mu^0 \left[D_0\Lambda\right]^a - \delta_\mu^i \left[D_i\Lambda\right]^a = -\left[D_\mu\Lambda\right]^a , \tag{29}$$

which is the well known gauge transformation of the theory. The generator of these transformations is given by

$$\mathcal{G}^g \equiv \mathcal{G}_a^\lambda \delta\lambda^a + \mathcal{G}_a^\omega \delta\omega^a = -\left(\mathcal{G}_a^\lambda D_0^{ab} + \delta^{ab}\mathcal{G}_a^\omega\right)\Lambda_b , \tag{30}$$

which is checked by computing

$$\delta A_\mu^a = \left\{A_\mu^a, G^g\right\} = -\left[D_\mu \Lambda\right]^a . \tag{31}$$

IV. THE HJ ANALYSIS ON THE NULL-PLANE

It was first pointed out by Dirac[28] that the dynamics given by the evolution of Cauchy surfaces of constant $t = x^0$ is not the only form of hamiltonian dynamics for relativistic theories. A good choice of hamiltonian dynamics implies the definition of a "time" axis and a family of hyper-surfaces with two properties: (1) the family must be orthogonal to the time axis in every point of the space-time, and (2) the path of a point particle cannot cross a member of the family more than once.

In classical mechanics there is only one hamiltonian dynamics. Given the fact that the classical "galilean time" is the same for all observers, and that particles can travel at any speed, the only family of hyper-surfaces that obey both conditions stated above is the family of euclidian spaces labelled by members of the real line. In this case, we may say that the galilean four dimensional space-time is decomposed in the form $\mathbb{R} \times \mathbb{E}^3$. In special relativity this is not the case. The background is now a Minkowski space-time $\mathbb{M}^d$, where d is the dimension, with a pseudo-euclidian metric. Because the path of a point-particle is bounded inside the light-cone, the instant-form dynamics, characterised by the decomposition $\mathbb{M}^d = \mathbb{R} \times \mathbb{E}^{d-1}$, used in Sec. III, is not unique. In fact, there are five forms of distinct relativistic dynamics.[29]

We are interested, in this section, in the null-plane dynamics whose evolution is given by the hyper-surfaces orthogonal to the axis $\tau = x^+ \equiv (x^0 + x^2)/\sqrt{2}$, in 2 + 1 dimensions. The axis itself belongs to the light-cone, so the orthogonal hyper-surfaces are characteristic planes, called null-planes. A remarkable feature is that regular theories become constrained on the null-plane dynamics which, in general, leads to a reduction in the number of independent field operators in the respective phase space due to the presence of extra non-involutive constraints.[30,31] The HJ formalism in the null-plane dynamics has been studied in the case of complex scalar and electromagnetic fields,[24] and also in linearised gravity.[25]

The coordinates of the light-cone are the most convenient for the analysis of field theories on the null-plane. In 2 + 1 dimensions these coordinates are defined by

$$x^+ \equiv \frac{1}{\sqrt{2}}\left(x^0 + x^2\right), \quad x^- \equiv \frac{1}{\sqrt{2}}\left(x^0 - x^2\right), \quad x^1 = x^1. \tag{32}$$

The time evolution is considered along the $\tau \equiv x^+$ coordinate, so a field configuration evolves from a characteristic 2-surface Σ_{τ_0} to a later surface Σ_{τ_1}, where Σ_τ is a null-plane defined by $\tau =$ *constant*. Moreover, the initial value problem that defines the hamiltonian dynamics on null-planes is not a Cauchy problem, but a characteristic value problem instead. This means simply that a unique evolution is not assured by the knowledge of a configuration of the fields and their velocities in a single initial time surface, but now by the values of the fields over two characteristic surfaces $\Sigma_{x^+=x_0^+}$ and $\Sigma_{x^-=x_0^-}$. This problem, however, does not affect our considerations.

The conjugated momenta are now given by

$$\pi_a^\mu \equiv \frac{\partial \mathcal{L}}{\partial \partial_+ A_\mu^a} = -F^{a+\mu} + \frac{\mu}{2}\varepsilon^{+\mu\nu} A_\nu^a. \tag{33}$$

The Levi-Civita symbol has now the value of $\varepsilon^{+\,-\,1} = 1$. Therefore, we have the following expressions:

$$\pi_a^+ = 0, \tag{34a}$$

$$\pi_a^- = F_{a+-} + \frac{\mu}{2} A_{a1}, \tag{34b}$$

$$\pi_a^1 = F_{a-1} - \frac{\mu}{2} A_{a-}. \tag{34c}$$

We notice that (34a) and (34c) are canonical constraints, which is the first difference compared with the instant-form dynamics, where we had a single set of constraints given by $\pi_a^0 = 0$.

Equation (34b) is a dynamical relation, resulting in the expression

$$\partial_+ A_-^a = \pi^{a-} + \partial_- A_+^a - g f^a_{\ bc} A_+^b A_-^c - \frac{\mu}{2} A_-^a. \tag{35}$$

The canonical hamiltonian density is given by

$$\mathcal{H}_c = \frac{1}{2}\left[\pi_a^- - \frac{\mu}{2} A_{a1}\right]^2 - A_+^a \left[\left[D_- \pi^-\right]_a + \left[D_1 \pi^1\right]_a - \frac{\mu}{2}\left(\partial_1 A_{a-} - \partial_- A_{a1}\right)\right]. \tag{36}$$

This way, we have the set of HJPDEs

$$\mathcal{H}' \equiv p_+ + \mathcal{H}_c = 0 \quad \rightarrow \quad \tau = x^+, \tag{37a}$$

$$\mathcal{H}_a'^+ \equiv \pi_a^+ = 0 \quad \rightarrow \quad \lambda_+^a \equiv A_+^a, \tag{37b}$$

$$\mathcal{H}_a'^1 \equiv \pi_a^1 - F_{-1}^a + \frac{\mu}{2} A_-^a = 0 \quad \rightarrow \quad \lambda_1^a \equiv A_1^a. \tag{37c}$$

The variables x^+, λ_+^a, and λ_1^a are the parameters of the theory, and each one is related to a hamiltonian density. On the other hand, the Poisson bracket structure remains the same: $\left\{A_\mu^a(x), \pi_b^\nu(y)\right\} = \delta_b^a \delta_\mu^\nu \delta(x-y)$. The IC is studied with the fundamental differential

$$dF = \int d^3y \left[\left\{F, \mathcal{H}'(y)\right\}^* dx^+ + \left\{F, \mathcal{H}_a'^+(y)\right\}^* d\lambda^a(y) + \left\{F, \mathcal{H}_a'^1(y)\right\}^* d\omega^a(y)\right].$$

We see that $\mathcal{H}_a'^+$ is in involution with $\mathcal{H}_a'^1$, but not with $\mathcal{H}'$. When imposing $d\mathcal{H}_a'^+ = 0$, the IC leads us to define a new hamiltonian density

$$\mathcal{C}_a' \equiv \left[D_- \pi^-\right]_a + \left[D_1 \pi^1\right]_a - \frac{\mu}{2}\left(\partial_1 A_{a-} - \partial_- A_{a1}\right) = 0. \tag{38}$$

On the other hand, we have that $\mathcal{H}_a'^1$ is a non-involutive set, since

$$\left\{\mathcal{H}_a'^1(x), \mathcal{H}_b'^1(y)\right\} = -2\left[D_-^x\right]_{ab} \delta(x-y), \tag{39}$$

where $\left[D_-^x\right]_{ab} \equiv \delta_{ab}\partial_-^x - g f_{abc} A_-^c(x)$. We also have $\left\{\mathcal{H}_a'^1(x), \mathcal{H}'(y)\right\} \neq 0$.

Following the procedure outlined in Ref. 20, we define a matrix

$$M_{ab}(x,y) \equiv \left\{\mathcal{H}_a'^1(x), \mathcal{H}_b'^1(y)\right\},$$

and calculate its inverse, $G_{ab}(x,y) \equiv M_{ab}^{-1}(x,y)$, which satisfies

$$\int d^3z M_{ac}(x,z) G^{cb}(z,y) = \delta_a^b \delta(x-y),$$

or explicitly, by using (39),

$$\left[D_-^x\right]_{ac} G^{cb}(x,y) = -\frac{1}{2}\delta_a^b \delta(x-y). \tag{40}$$

Notice that this equation does not depend on the topological mass μ, then it is also valid for the free Yang-Mills field. Given the inverse $G_{ab}(x, y)$ we build the GB

$$\begin{aligned}\{F(x), G(y)\}^* \equiv{}& \{F(x), G(y)\} \\ &- \int dzdw \left\{F(x), \mathcal{H}_a'^1(y)\right\} G^{ab}(z,w) \left\{\mathcal{H}_b'^1(x), G(y)\right\}.\end{aligned} \tag{41}$$

Using this expression and (40) we obtain the following fundamental GBs

$$\left\{A_\mu^a(x), A_\nu^b(y)\right\}^* = \delta_\mu^1 \delta_\nu^1 G^{ab}(x,y), \tag{42a}$$

$$\left\{A_\mu^a(x), \pi_b^\nu(y)\right\}^* = \delta_b^a \delta_\mu^\nu \delta(x-y)$$
$$-\delta_\mu^1 \left\{\delta_1^\nu \left[D_-^y\right]^{ac} - \delta_-^\nu \left[D_1^y\right]^{ac} + \frac{\mu}{2}\delta^{ac}\delta_-^\nu\right\} G_{cb}(x, y), \tag{42b}$$

$$\left\{\pi_a^\mu(x), \pi_b^\nu(y)\right\}^* = \left\{\delta_1^\nu \left[D_-^y\right]_{ac} - \delta_-^\nu \left[D_1^y\right]_{ac} + \frac{\mu}{2}\delta_{ac}\delta_-^\nu\right\} \times$$
$$\times \left\{\delta_1^\mu \left[D_-^x\right]^{cd} - \delta_-^\mu \left[D_1^x\right]^{cd} + \frac{\mu}{2}\delta^{cd}\delta_-^\mu\right\} G_{db}(x, y). \tag{42c}$$

The GB between $\mathcal{H}_a'^1$ and any other function of the phase-space is identically zero, therefore the parameters λ_1^a are eliminated by the redefinition of the dynamics with (42). It can be shown that $\mathcal{H}_a'^+$ are involutive hamiltonians, since (42b) implies that the $\mathcal{H}_a'^+$ has non zero GB only with A_+^a. For $\mathcal{C}_a'$ we have

$$\left\{\mathcal{C}_a'(x), \mathcal{C}_b'(y)\right\}^* = g f_{ab}{}^c \mathcal{C}_c'(x)\,\delta(x-y),$$

therefore all hamiltonians are in involution with the GBs.

A. Characteristic equations

We have obtained the complete set of hamiltonians $\mathcal{H}'$, $\mathcal{G}_a^\lambda \equiv \mathcal{H}_a'^+$, and $\mathcal{G}_a^\omega \equiv \mathcal{C}_a'$, and each one is related to the parameters x^+, $\lambda^a \equiv \lambda_+^a$, and ω^a. This late set is introduced by expanding the phase-space such that the fundamental differential is now given by

$$dF = \int d^3y \left[\left\{F, \mathcal{H}'(y)\right\}^* dx^+ + \left\{F, \mathcal{G}_a^\lambda(y)\right\}^* d\lambda^a(y) + \left\{F, \mathcal{G}_a^\omega(y)\right\}^* d\omega^a(y)\right]. \tag{43}$$

By replacing $F = A_\mu^a(x)$ we obtain the first set of characteristic equations

$$dA_+^a = d\lambda^a, \tag{44a}$$

$$dA_-^a = \left[\pi^{a-} + [D_- A_+]^a - \frac{\mu}{2}A_1^a\right] dx^+ - [D_- d\omega]^a, \tag{44b}$$

$$dA_1^a = dx^+ \int d^3y \left\{\left[D_1^y\right]^{ab} - \delta^{ab}\frac{\mu}{4}\right\} G_{bc}(x, y)\, F_{+-}^c(y) + [D_1 A_+]^a dx^+ - [D_1 d\omega]^a. \tag{44c}$$

Considering the time evolution by itself, we see that Eq. (44a) indicates that A_+^a still remains as a parameter of the theory. Equation (44b) is equivalent to the dynamical relation (35), which results from the definition of the canonical momenta π_a^-. To obtain the remaining field equations, for the A_1^a variables, we simply apply D_-^x in (44c). Therefore, full equivalence between the characteristic equations and the field equations is assured.

B. Generator of the canonical and gauge transformations

The evolution of the variables A_μ^a along the parameters λ^a and ω^a is given by

$$\delta A_\mu^a = \delta_\mu^+ \delta\lambda^a - \delta_\mu^- [D_- \delta\omega]^a - \delta_\mu^1 [D_1 \delta\omega]^a. \tag{45}$$

From this expression, we identify the generator of canonical transformations

$$G^{can} \equiv \int dy \left(\mathcal{G}_a^\lambda \delta\lambda^a + \mathcal{G}_a^\omega \delta\omega^a\right) \tag{46}$$

and (45) is recovered by the expression

$$\delta A_\mu^a = \left\{A_\mu^a, G^{can}\right\}^*. \tag{47}$$

As we have done in Sec. III B, we may obtain the generator of the gauge transformation by considering the set of variations (45) as symmetries of the TMYM Lagrangian density. In the null-plane dynamics, the fixed point variation is written as

$$\delta\mathcal{L} = \left(\delta^+_\alpha \delta\lambda^a_+ - \delta^-_\alpha \left[D_- \delta\omega\right]^a - \delta^1_\alpha \left[D_1 \delta\omega\right]^a\right)\left(\left[D_\nu F^{\nu\alpha}\right]_a + \frac{\mu}{2}\varepsilon^{\alpha\beta\gamma} F_{a\beta\gamma}\right)$$
$$-\partial_\mu \left[\delta A^a_\alpha \left(F^{\mu\alpha}_a + \frac{\mu}{2}\varepsilon^{\mu\alpha\gamma} A_{a\gamma}\right)\right].$$

If we eliminate boundary terms and rearrange this equation, we obtain

$$\delta\mathcal{L} = F^{a+-}\left[D_-\right]_{ab}\left(\delta\lambda^b_+ + \left[D_+\delta\omega\right]^b\right) + F^{a+1}\left[D_1\right]_{ab}\left(\delta\lambda^b_+ + \left[D_+\delta\omega\right]^b\right)$$
$$+\mu\left[\delta\lambda^a_+ F_{a-1} + \delta\omega^a\left(\left[D_- F_{1+}\right]_a + \delta\omega^a\left[D_1 F_{+-}\right]_a\right)\right]. \tag{48}$$

In the three dimensional null-plane coordinates, the Bianchi identities become

$$\left[D_+ F_{-1}\right]^a + \left[D_1 F_{+-}\right]^a + \left[D_- F_{1+}\right]^a = 0,$$

then we may write (48) as

$$\delta\mathcal{L} = \left[F^{a+-}\left[D_-\right]_{ab} + F^{a+1}\left[D_1\right]_{ab} + \mu\delta_{ab}F^a_{-1}\right]\left(\delta\lambda^b_+ + \left[D_+\delta\omega\right]^b\right).$$

Invariance of the theory under the transformations (45) yields $\delta L = 0$. In this case, we have $\delta\lambda^a_+ = -\left[D_+\delta\omega\right]^a$. Under this condition, and choosing $\delta\omega^a = \Lambda^a$, (45) now turns to be

$$\delta A^a_\mu = -\delta^+_\mu\left[D_+\delta\omega\right]^a - \delta^-_\mu\left[D_-\delta\omega\right]^a - \delta^1_\mu\left[D_1\delta\omega\right]^a = -\left[D_\mu\Lambda\right]^a, \tag{49}$$

which are the correct gauge transformations. Furthermore, the generator is written as

$$G^g \equiv \int dy\left(-\mathcal{G}^\lambda_a\left[D_+\right]^{ab} + \delta^{ab}\mathcal{G}^\omega_a\right)\Lambda_b, \tag{50}$$

since we have

$$\delta A^a_\mu = \left\{A^a_\mu, G^g\right\} = -\left[D_\mu\Lambda\right]^a. \tag{51}$$

V. FINAL REMARKS

We have analysed the constraint structure of the TMYM theory in the instant-form and null-plane dynamics using the HJ formalism. We found the complete set of hamiltonians that generate the dynamical evolution, established the characteristic equations and the equivalence between these and the field equations, and studied the set of canonical transformations that are symmetries of the system, which gives us the generator of gauge transformations.

In instant-form this analysis is straightforward, since all constraints are involutive ones. The hamiltonian densities $\mathcal{H}'^0_a$, which come from the definition of the canonical momenta, are in involution among themselves. But they are not in involution with $\mathcal{H}'$, which is the constraint that involves the canonical hamiltonian density (16). The Poisson brackets between $\mathcal{H}'$ and $\mathcal{H}'^0_a$ give rise to another set of hamiltonians $\mathcal{C}'_a = 0$, Eq. (18). With the complete set of involutive hamiltonians of the system the dynamics in instant-form is given by the fundamental differential (21), and the characteristic equations of the system, (22), follow. The temporal part of the dynamics is shown to be equivalent to the field equation, while the dynamics along the parameters λ^a and ω^a are considered as canonical transformations whose generator is given by (26). Building the generator of gauge transformations only requires to find the conditions in which the action is invariant, resulting in the generator given by (30).

In the null-plane dynamics, on the other hand, we have a larger number of HJ equations that come from the definition of the canonical momenta. They define the hamiltonians $\mathcal{H}'$ and $\mathcal{H}'^+_a$, Eqs. (37a) and (37b), which resemble the ones found in instant-form, but the constraint structure has also the presence of the hamiltonians $\mathcal{H}'^1_a = 0$, (37c). Integrability of the hamiltonians $\mathcal{H}'^+_a$ again results in the definition of $\mathcal{C}'_a = 0$, (38). However, $\mathcal{H}'^1_a$ are not in involution either with $\mathcal{H}'$ or with themselves. Their integrability conditions imply that their respective parameters λ^1_a are dependent of the others. The procedure used to get rid of this dependence was developed in Ref. 20, and requires the analysis of the matrix M_{ab} defined in (39). In this case we have that the generalized brackets (41) must now replace the PB of the theory.

With the GB, the fundamental differential is written in the form (43), giving rise to the characteristic equations (44) of the system. Again, time evolution alone is shown to be equivalent to the field equations. On the other hand, the dynamics in the direction of the parameters λ^a and ω^a, (45), which are canonical transformations in the fields, can be related to gauge transformations imposing invariance to the Lagrangian density. In this case, it results that the generator of gauge transformations of the theory is given by (50).

Finally, let us remark that the Hamilton-Jacobi constraint analysis does not need any pre-defined gauge condition. This is in contrast with the Dirac's hamiltonian formalism presented in Ref. 10, where a gauge is fixed at the classical level, specifically the null axial gauge $A^a_- = 0$, for the TMYM theory in the null-plane dynamics. Imposing a gauge in the HJ formalism would spoil the construction of the generator (50). A detailed analysis of the construction of gauge generators using the HJ formalism will be shown in Ref. 27.

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