The canonical structure of Podolsky's generalized electrodynamics on the Null-Plane

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In this work we will develop the canonical structure of Podolsky's generalized electrodynamics on the null-plane. This theory has second-order derivatives in the Lagrangian function and requires a closer study for the definition of the momenta and canonical Hamiltonian of the system. On the null-plane the field equations also demand a different analysis of the initial-boundary value problem and proper conditions must be chosen on the null-planes. We will show that the constraint structure, based on Dirac formalism, presents a set of second-class constraints, which are exclusive of the analysis on the null-plane, and an expected set of first-class constraints that are generators of a $U(1)$ group of gauge transformations. An inspection on the field equations will lead us to the generalized radiation gauge on the null-plane, and Dirac Brackets will be introduced considering the problem of uniqueness of these brackets under the chosen initial-boundary condition of the theory.

1. Introduction

Most physical systems, including fundamental fields in quantum field theory, are described by Lagrangians that depend at most on first-order derivatives. However, there is a continuous interest on theories with higher-order derivatives, either do accomplish generalizations or to get rid of some undesirable properties of first-order theories. This interest had begun in the half of the 19th century, when Ostrogradski [1] developed the Hamiltonian formalism for this kind of system in classical mechanics.

As examples of systems treated by higher-order Lagrangians we mention the attempts to solve the problem of renormalization of the gravitational field by inserting quadratic terms of the Riemann tensor and its contractions [2,3,4] on the Einstein-Hilbert Action. Recent developments in this direction has been made by Cuzinatto *et al.* [5] where the construction of high-order Lagrangians for gravity is made with invariants of the Riemann tensor taking account the local Lorentz invariance. This attempt turns out to be a natural generalization of the Utiyama's Theory of General Gauge Fields [6] applied to second-order theories [7].

Higher-order Lagrangians have also emerged as effective theories on the infrared sector of the QCD [8], where it enforces a good asymptotic behavior of the gluon propagator. It is also important to remark that the inclusion of higher-order derivatives in field theory of supersymmetric fields has shown to be a powerful regularization mechanism [9,10]. We noted that a very attractive property of quantum field theories with higher-order terms is the fact that it improves the convergence of the corresponding Feynman diagrams[11,12].

The first model of a higher-order derivative field theory is a generalization of the electromagnetic field proposed in the works of Podolsky, Schwed and Bopp [13,14], which culminated in the Podolsky's generalized electrodynamics. It is suggested to modify the Maxwell-Lorentz theory in order to avoid divergences such as the electron self-energy and the vacuum polarization current. These difficulties can be traced to the fact that the classical electrodynamics involve an r^{-1} singularity that results in an infinite value of the electron self-energy. The Lagrangian density is, therefore, modified by a second-order derivative term:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}a^2\partial_\lambda F^{\mu\lambda}\partial^\gamma F_{\mu\gamma} \ , \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{1}$$

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Podolsky's theory already has many interesting features at the classical level. It solves the problem of infinite energy in the electrostatic case and also gives the correct expression for the self-force of charged particles at short distances, as showed by Frenkel [15], solving the problem of the singularity at $\mathbf{r} \to 0$. It has been shown by Cuzinatto *et al.* [16] that the above Lagrangian density is the only possible generalization of the electromagnetic field that preserves invariance under $U(1)$. Besides, the theory yields field equations that are still linear in the fields.

Another important prediction of the model is the existence of massive photons, whose mass is proportional to the inverse of the Podolsky's parameter a. This feature allows experiments that may test the generalized electrodynamics as a viable effective theory. The determination of an upper bound value for the mass of the photon is actually a current concern in the theoretical framework [16].

The canonical quantization of the field was tried in the work of Podolsky and Schwed [13]. However, Podolsky's theory inherits the same difficulties from the standard electromagnetic field, the presence of a degenerate variable, which forced them to use a Fermi-like Lagrangian. However, the chosen gauge fixing condition, the usual Lorenz condition, does not fulfill the requirements for a good choice of gauge in the context of Podolsky's theory. The first consistent approach to the quantization of the field was given by Galvão and Pimentel [17], where Dirac canonical formalism [18,19,20] is used with the correct choice of gauge.

The first attempts of quantization of Podolsky's field was made in instant-form, where the "laboratory time" $t = x^0$ is the evolution parameter of the theory. Dirac [21] was the first to notice that this choice is not the only possible parametrization for field theories. Actually it is possible to define five different forms of Hamiltonian dynamics, each one related to different sub-groups of the Poincaré group [22]. In this work we intend to proceed with the canonical analysis in front-form dynamics, also called the null-plane parametrization, where the coordinate $x^+ \equiv 1/\sqrt{2}\left(x^0 + x^3\right)$ is chosen to be the evolution parameter. This parameter choice was mistaken, for some time, to the so called infinite-momentum frame [23], which is a limit process to analyse field theories in a frame near to the speed of light. It is a choice of coordinate system rather than a physical reference frame, and implies the definition of the null-plane which is the 3-surface $x^+ = 0$. The classical evolution of the system is then given by the definition of appropriate brackets plus a set of initial data, which are the configuration of the fields at the above 3-surface.

The paper is organized as follows. In section two we discuss the null-plane coordinates, which are the natural coordinate system for the front-form dynamics, and we review the initial-boundary value problem for the fields and establish appropriate conditions to achieve a unique solution of the dynamic equations on the null-plane. Section three will be devoted to a review on the Hamiltonian formalism for higher-order Lagrangians. In Section four the canonical approach is applied for the generalized electromagnetic field in null-plane coordinates. In section five we establish a set of consistent gauge conditions and corresponding Dirac Brackets to describe the physical dynamics of the theory. Section six will be devoted for the final remarks.

2. The null-plane coordinates

As the start-point for the analysis of a field theory we have the Action

$$S[\phi] \equiv \int_\Omega d\omega \; \mathcal{L}\left(\phi, \partial_\mu \phi\right) \;,$$

where $\mathcal{L}$ is a Lagrangian density, and $d\omega$ is a four-volume element of a finite (or usually infinite) four-volume Ω of the space-time. For relativistic theories the Lagrangian density must be chosen to be invariant under any particular parameter choice. However, although the Lagrangian formalism preserves this invariance, the same does not occur in the Hamiltonian formalism, which requires a parametrization in order to be fully carried out.

Dirac has shown [21] that the usual dynamics, the instant-form, where the galilean time $x^0 = t$ is the parameter that defines the evolution of the system from a given initial 3-surface $\Gamma_{t=t_0}$ to a later surface $\Gamma_{t=t_1}$, is not the only possible choice of parametrization. He calls attention for two other forms of Hamiltonian dynamics: the punctual-form and the front-form. Later, two other forms were discovered [24].

An important advantage pointed out by Dirac is the fact that seven of the ten Poincaré generators are kinematical on the null-plane while the conventional theory constructed in instant-form has only six of these generators. Therefore, the structure of the phase space is distinct in both cases. As such, a description of the physical systems on the null-plane could give additional information from those provided by the conventional formalism [25]. Another remarkable feature is that regular theories become constrained when analyzed on the null-plane. In general, it leads to a reduction in the number of independent field operators in the respective phase space due to the presence of second-class constraints.

The natural coordinate system of instant-form dynamics is the rectangular system $x^\mu \equiv \left(x^0, x^1, x^2, x^3\right)$. We can pass to null-plane coordinates with the linear transformation $x' = \Sigma\, x$ where the transformation matrix and its inverse are given by

$$\Sigma \equiv \frac{1}{\sqrt{2}} \left(\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} \cdot \mathbf{I} & 0 \end{array} \right) , \qquad \Sigma^{-1} = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \cdot \mathbf{I} \\ 1 & -1 & 0 \end{array} \right) ,$$

where $\mathbf{I}$ is the 2×2 identity matrix and $x'^\mu \equiv \left(x^+, x^-, x^1, x^2\right)$.

Lorentz tensors are also covariant under this transformation, but the transformation itself is not of Lorentz type [25]: if in usual coordinates we define the Minkowski metric as $\eta \equiv$diag$(1, -1, -1, -1)$, the metric in null-plane coordinates will be given by

$$\eta' = \Sigma\, \eta\, \Sigma^{-1} = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\mathbf{I} \end{array} \right) .$$

Using this metric (from now on we will ignore the comma) we can see that the norm of a vector is not a quadratic form, but will be linear in the longitudinal components.

Of special interest is the D'Alambertian operator

$$\Box \equiv \partial_\mu \partial^\mu = 2\partial_- \partial_+ + \partial_i \partial^i \ . \tag{2}$$

Since the evolution parameter is x^+ a field equation like $\left(\Box + m^2\right)\phi = 0$ will be linear on the velocity $\partial_+\phi$, which does not occur in instant-form. Therefore, the analysis of initial-boundary value problem is changed from a Cauchy to a characteristics initial-boundary value problem. This is due to the fact that a quadratic Lagrangian on $\partial_0\phi$ is actually of first-order on $\partial_+\phi$. In the case of the scalar field on the null-plane it is sufficient to fix the values of the fields on both characteristics surfaces to solve the field equations [26,27,28].

This can be seen in Podolsky's case by the Euler-Lagrange (EL) equations of the Lagrangian (1):

$$\left(1 + a^2\Box\right)\Box A_\mu - \partial_\mu \left(1 + a^2\Box\right)\partial^\nu A_\nu = 0 \ . \tag{3}$$

This equation is fourth-order in $\partial_0 A_\mu$ but only second-order in $\partial_+ A_\mu$. Therefore, in instant-form it is necessary to specify four conditions, the values of the field and of its derivatives until third-order on an initial surface $x^0 = 0$ to uniquely write a solution.

On the null-plane the equation is just of second-order, but the existence of two characteristics surfaces demands the knowledge of four initial-boundary conditions as well. The normal vector of a null-plane lies in the same plane, therefore, the knowledge of the value A_μ on a null-plane implies in its normal derivative $\partial_+ A_\mu$. Thus, the solution of the field equations is uniquely determined if A is specified on the null-plane $x^+ = cte$ and three boundary conditions are imposed on $x^- = cte$, which in our case consists on the value of the derivatives of the field up to third-order.

In the canonical framework, it was Steinhardt [30] who showed that to linear Lagrangians the initial condition on $x^+ = cte$ plus a Hamiltonian function are insufficient to predict uniquely all physical process. Boundary conditions along the $x^- = cte$ plane must also be determined. He also observed that the matrix formed by the Poisson Brackets (PB) of the second-class constraints does not have a unique inverse and that the presence of arbitrary functions is associated with the insufficiency of the initial value data. It is also responsible for the existence of a hidden subset of first-class constraints which is associated with improper gauge transformations [29]. By imposing appropriate initial-boundary conditions on the fields, the hidden first-class constraints can be eliminated in order to the total Hamiltonian be a true generator of the physical evolution. It will also determine an unique inverse of the second class constraint matrix which allows to obtain the correct Dirac Brackets among the fundamental variables. Thus, in the study of the Podolsky's theory we follow the same tune outlined in [31,32,33].

3. Second-order derivatives on the null-plane

Let us consider a generic Lagrangian density $\mathcal{L}(\phi,\partial\phi,\partial^2\phi)$ dependent of a number n of fields $\phi^a(x)$ and its first and second derivatives. The application of Hamilton's Action principle yields the following EL equations

$$\frac{\delta\mathcal{L}}{\delta\phi^a}-\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi^a\right)}\right]+\partial_\mu\partial_\nu\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\partial_\nu\phi^a\right)}\right]=0\ ,$$

which are the equations that originates (3) from (1). On the solutions, the vanishing of the variation of the action yields the conserved symmetric energy-momentum tensor

$$\begin{aligned}T_{\mu\nu}\ \ \equiv\ \ &\partial_\mu\phi^a\ \frac{\delta\mathcal{L}}{\delta\left(\partial^\nu\phi^a\right)}-\mathcal{L}\eta_{\mu\nu}\\&-2\partial_\mu\phi^a\partial_\lambda\left[\frac{\delta\mathcal{L}}{\delta\left(\partial^\nu\partial_\lambda\phi^a\right)}\right]+\partial_\lambda\left[\partial_\mu\phi^a\ \frac{\delta\mathcal{L}}{\delta\left(\partial^\nu\partial_\lambda\phi^a\right)}\right]-\partial^\lambda\left(\Xi_{\mu\lambda\nu}+\Pi_{\mu\lambda\nu}\right)\ ,\end{aligned}\tag{4}$$

where

$$\begin{aligned}\Xi_{\mu\lambda\nu}\ \ &\equiv\ \ \frac{1}{2}\left[\frac{\delta\mathcal{L}}{\delta\left(\partial^\mu\phi^a\right)}-\partial_v\left(\frac{\delta\mathcal{L}}{\delta\left(\partial^\mu\partial_\nu\phi^a\right)}\right)\right]\left(\mathbf{I}_{\lambda\nu}\right)^a{}_b\ \phi^b\ ,\\\Pi_{\mu\lambda\nu}\ \ &\equiv\ \ \frac{1}{2}\frac{\delta\mathcal{L}}{\delta\left(\partial^\mu\partial^\alpha\phi^a\right)}\ \left(\mathbf{I}_{\lambda\nu}\right)^a{}_b\ \partial^\alpha\phi^b\ .\end{aligned}$$

$\left(\mathbf{I}_{\lambda\nu}\right)^a{}_b$ are the infinitesimal generators of the Poincaré group.

The conserved charge is given by the expression

$$G\equiv-a^\mu P_\mu-\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu}$$

with generators

$$\begin{aligned}P_\mu\ \ &\equiv\ \ \int_\sigma d\sigma^\nu T_{\mu\nu}\ ,\\M_{\mu\nu}\ \ &\equiv\ \ \int_\sigma d\sigma^\alpha\left(T_{\alpha\mu}x_\nu-T_{\alpha\nu}x_\mu\right)\ .\end{aligned}$$

In the above expressions σ is a 3-surface orthogonal to the parametrization axis.

If we choose the null-plane, we will be interested in the dynamical generator P_+, which is given by

$$P_+\equiv\int d^3xT_{+-}$$

where we adopt $d^3x\equiv dx^-dx^1dx^2$. Here we have the canonical Hamiltonian density

$$\begin{aligned}\mathcal{H}_c\ \ \equiv\ \ T_{+-}=&\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_+\phi^a\right)}-\partial_+\ \frac{\delta\mathcal{L}}{\delta\left(\partial_+\partial_+\phi^a\right)}-2\partial_-\ \frac{\delta\mathcal{L}}{\delta\left(\partial_-\partial_+\phi^a\right)}-2\partial_i\ \frac{\delta\mathcal{L}}{\delta\left(\partial_i\partial_+\phi^a\right)}\right]\partial_+\phi^a\\&+\partial_+\partial_+\phi^a\ \frac{\delta\mathcal{L}}{\delta\left(\partial_+\partial_+\phi^a\right)}-\mathcal{L}\ .\end{aligned}\tag{5}$$

This result suggests the following definition for the canonical momenta:

$$p_a\ \ \equiv\ \ \int d^3x\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_+\phi^a\right)}-\partial_+\ \frac{\delta\mathcal{L}}{\delta\left(\partial_+\partial_+\phi^a\right)}-2\partial_-\ \frac{\delta\mathcal{L}}{\delta\left(\partial_-\partial_+\phi^a\right)}-2\partial_i\ \frac{\delta\mathcal{L}}{\delta\left(\partial_i\partial_+\phi^a\right)}\right]\ ,\tag{6a}$$

$$\pi_a\ \ \equiv\ \ \int d^3x\frac{\delta\mathcal{L}}{\delta\left(\partial_+\partial_+\phi^a\right)}\ ,\tag{6b}$$

where the fields ϕ and $\partial_+\phi$ are treated as independent canonical fields.

It is straightforward to show that the EL equations can be written by

$$W_{ab}\left(\partial_+\right)^4\phi^b=F_a\left(\phi,\partial\phi,\partial^2\phi,\partial^3\phi\right)$$

where the generalized Hessian matrix is

$$W_{ab} \equiv \frac{\delta \pi_a}{\delta\left(\partial_+\partial_+\phi^b\right)} = \int d^3x \frac{\delta \mathcal{L}}{\delta\left(\partial_+\partial_+\phi^a\right)\delta\left(\partial_+\partial_+\phi^b\right)} \; . \tag{7}$$

It is the regularity or the singularity of this matrix that determines the regularity or the singularity of the system.

In this analysis we have ignored the boundary conditions of the fields, which is a quite misleading attitude, since the null-plane dynamics requires a different analysis of initial-boundary conditions than the instant-form dynamics. The discussion about the initial-boundary value problem in this case will be made properly during the canonical procedure, so at this point we just make sure that the conditions of the fields are equivalent of those in instant-form, in other words, that the fields and all required derivatives go to zero at the boundary of the 3-surface.

4. The Hamiltonian analysis

From the Lagrangian density (1) and the definitions (6) follows the canonical momenta for the Podolsky's field

$$p^\mu = F^{\mu+} - a^2\left(\eta^{\mu-}\partial_-\partial_\lambda F^{+\lambda} + \eta^{\mu i}\partial_i\partial_\lambda F^{+\lambda} - 2\partial_-\partial_\lambda F^{\mu\lambda}\right) \; , \tag{8a}$$

$$\pi^\mu = a^2\eta^{\mu+}\partial_\lambda F^{+\lambda} \; . \tag{8b}$$

The Hessian matrix of this system is

$$W^{\mu\nu} = \frac{\delta \pi^\mu}{\delta\left(\partial_+\partial_+A_\nu\right)} = -a^2\eta^{\mu+}\delta^\nu_-\eta^{++} = 0 \; .$$

As we saw in the earlier section the fields A_μ and $\partial_+ A_\mu$ should be treated as independent variables. Therefore we will use the notation $\bar{A}_\mu \equiv \partial_+ A_\mu$, being A_μ and $\bar{A}_\mu$ independent fields. Then we are able to define the primary constraints

$$\phi_1 = \pi^+ \approx 0 \; , \tag{9a}$$

$$\phi_2^i = \pi^i \approx 0 \; , \tag{9b}$$

$$\phi_3 = p^+ - \partial_-\pi^- \approx 0 \; , \tag{9c}$$

$$\phi_4^i = p^i - \partial_i\pi^- + F_{i-} + 2a^2\partial_-\left[\partial_i\bar{A}_- - 2\partial_-\bar{A}_i + \partial_i\partial_-A_+ - \partial_jF_{ij}\right] \approx 0 \; . \tag{9d}$$

The canonical Hamiltonian density can be expressed by

$$\begin{aligned}\mathcal{H}_c = {} & p^\mu\bar{A}_\mu + \pi^-\left(\partial_-\bar{A}_+ - \partial^i\bar{A}_i + \partial^i\partial_iA_+\right) - \frac{1}{2}\left(\bar{A}_- - \partial_-A_+\right)^2 - \left(\bar{A}_i - \partial_iA_+\right)F_{-i} \\ & + \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}a^2\left(\partial_i\bar{A}_- - 2\partial_-\bar{A}_i + \partial_i\partial_-A_+ - \partial_jF_{ij}\right)^2 \; .\end{aligned} \tag{10}$$

With the canonical Hamiltonian $H_c = \int d^3x\mathcal{H}_c(x)$ and the primary constraints (9) we build the primary Hamiltonian

$$H_P \equiv H_c + \int d^3x u^a(x)\phi_a(x) \; , \qquad \{a\} = \{1,2,3,4\} \; . \tag{11}$$

To proceed with the calculus of the consistency conditions we use the primary Hamiltonian as generator of the x^+ evolution and define the fundamental equal x^+ Poisson Brackets with the expressions

$$\left\{A_\mu(x), p^\nu(y)\right\}_{x^+=y^+} = \left\{\bar{A}_\mu(x), \pi^\nu(y)\right\}_{x^+=y^+} = \delta^\nu_\mu\delta^3(x-y) \; , \tag{12}$$

where $\delta^3(x-y) \equiv \delta(x^- - y^-)\delta^2(\mathbf{x}-\mathbf{y})$. We verify that the condition $\dot{\phi}_1 \approx 0$ gives just the constraint $\phi_3 \approx 0$, which is already satisfied. The consistency for the remaining constraints gives equations for some Lagrange multipliers. Notice that the conditions for ϕ_2^i and ϕ_3,

$$\dot{\phi}_2^i = -\phi_4^i + 4a^2\partial_-\partial_-u_i^4 \approx 0 \; ,$$

$$\dot{\phi}_3 = \partial_-p^- + \partial_ip^i + 4a^2\partial_i\partial_-\partial_-u_i^4 \approx 0 \; ,$$

give equations for the same parameters u_i^4. These equations must be consistent to each other. From the first we have $\partial_-\partial_- u_i^4 \approx 0$ and, applying this result on the second condition, a secondary constraints appears:

$$\chi \equiv \partial_- p^- + \partial_i p^i \approx 0 \ .$$

For this secondary constraint, $\dot{\chi} = 0$, and no more constraints can be found.

The analysis leaves us with the following set:

$$\begin{aligned}
\chi &= \partial_- p^- + \partial_i p^i \approx 0 \ , \\
\phi_1 &= \pi^+ \approx 0 \ , \\
\phi_2^i &= \pi^i \approx 0 \ , \\
\phi_3 &= p^+ - \partial_- \pi^- \approx 0 \ , \\
\phi_4^i &= p^i - \partial_i \pi^- + F_{i-} + 2a^2 \partial_- \left[\partial_i \bar{A}_- - 2\partial_- \bar{A}_i + \partial_i \partial_- A_+ - \partial_j F_{ij} \right] \approx 0 \ .
\end{aligned}$$

It happens that χ and ϕ_1 are first-class constraints, while ϕ_2^i , ϕ_3 and ϕ_4^i are second-class ones. However, constructing the matrix of the second-class constraints we found that it is singular of rank four, which indicates that there must exist a first-class constraint, associated with the zero mode of this matrix, and its construction is made from the corresponding eigenvector which gives a linear combination of second-class constraints. The combination happens to be just $\Sigma_2 \equiv \phi_3 - \partial_i \phi_2^i$ and it is independent of χ and ϕ_1. Therefore, we have the renamed set of first-class constraints

$$\Sigma_1 \equiv \pi^+ \approx 0 \ , \tag{13a}$$
$$\Sigma_2 \equiv p^+ - \partial_- \pi^- - \partial_k \pi^k \approx 0 \ , \tag{13b}$$
$$\Sigma_3 \equiv \partial_- p^- + \partial_i p^i \approx 0 \ , \tag{13c}$$

and a set of irreducible second-class constraints

$$\Phi_1^i \equiv \pi^i \approx 0 \ , \tag{14a}$$
$$\Phi_2^i \equiv p^i - \partial_i \pi^- + F_{i-} + 2a^2 \partial_- \left[\partial_i \bar{A}_- - 2\partial_- \bar{A}_i + \partial_i \partial_- A_+ - \partial_j F_{ij} \right] \approx 0 \ . \tag{14b}$$

The second-class constraints do not appear in the instant-form dynamics for this theory: they are a common effect of the null-plane dynamics.

Here we are in position to write the total Hamiltonian

$$H_T \equiv H_c + \int d^3x u^a(x) \Sigma_a(x) + \int d^3x \lambda_i^I(x) \Phi_I^i(x) \ , \tag{15}$$

with which we are able to calculate the canonical equations of the system for the variables A_μ, $\bar{A}_\mu$, p^μ and π^μ.

For A_μ we have the equations

$$\partial_+ A_\mu = \bar{A}_\mu + \delta_\mu^+ u^2 - \delta_\mu^- \partial_- u^3 - \delta_\mu^i \left[\partial_i u^3 - \lambda_i^2 \right] \ , \tag{16}$$

which just means that the canonical variable $\bar{A}_\mu$ is defined as $\partial_+ A_\mu$ plus a linear combination of the still arbitrary Lagrange multipliers. The equations for $\bar{A}_\mu$ give

$$\partial_+ \bar{A}_\mu \approx \delta_\mu^+ u^1 + \delta_\mu^- \left[\partial_- \bar{A}_+ + \partial_i \bar{A}_i - \partial_i \partial_i A_+ + \partial_- u^2 + \partial_i \lambda_i^2 \right] + \delta_\mu^i \left[\partial_i u^2 + \lambda_i^1 \right] \ . \tag{17}$$

The equation for $\bar{A}_+$ is just $\partial_+ \bar{A}_+ \approx u^1$, which is expected since $\bar{A}_+$ is a degenerate variable. The expression for $\bar{A}_-$ can be written, using (16), as

$$\partial_\mu F^{-\mu} \approx - \left[\partial^i \partial_i + \partial^+ \partial_+ \right] u^3 \ . \tag{18}$$

The Hamiltonian equations for the momenta p^μ are given, with (16) and $\pi^- = a^2 \partial_\lambda F^{+\lambda}$, by

$$\begin{aligned}
\partial_+ p^+ &\approx \partial_\lambda F^{\lambda +} - a^2 \partial_i \partial_- \partial_\lambda F^{\lambda i} - a^2 \partial_i \partial_i \partial_\lambda F^{\lambda +} + \left(1 + a^2 \partial_i \partial_i \right) \partial_- \partial_- u^3 \ , \\
\partial_+ p^- &\approx \partial_i F^{i-} + \partial_i \partial_i u^3 \ , \\
\partial_+ p^i &\approx \partial_- F^{-i} + \partial_j F^{ji} - a^2 \partial_\mu \partial^\mu \partial_j F^{ij} - \partial_- \partial_i u^3 \ .
\end{aligned}$$

The equations for π^μ are, using the fact that π^+ and π^i are weakly zero,

$$\begin{aligned}
p^+ &\approx a^2\partial_-\partial_\lambda F^{+\lambda} \ , \\
p^- &\approx F^{-+} + a^2\partial_-\partial_\lambda F^{-\lambda} + \partial_- u^3 - a^2\partial_-\partial_i\partial_i u^3 \ , \\
p^i &\approx F^{i+} - a^2\left(\partial^i\partial_\lambda F^{+\lambda} - 2\partial_-\partial_\lambda F^{i\lambda}\right) + 2a^2\partial_-\partial_-\partial_i u^3 \ .
\end{aligned}$$

The last equations reproduce the definition of the canonical momenta p with some combination of the Lagrange multipliers. If we use these equation on the earlier equations for $\partial_+ p^\mu$, and also using (18), we have

$$\left(1+a^2\Box\right)\partial_\lambda F^{\lambda+} + \left(1+a^2\partial_i\partial_i\right)\partial_-\partial_- u^3 \approx 0 \ , \tag{19a}$$

$$\left(1+a^2\Box\right)\partial_\lambda F^{\lambda-} + a^2\partial_+\partial_-\partial_i\partial_i u^3 \approx 0 \ , \tag{19b}$$

$$\left(1+a^2\Box\right)\partial_\lambda F^{\lambda i} - \left(1+2a^2\partial_+\partial_-\right)\partial_-\partial_i u^3 \approx 0 \ . \tag{19c}$$

These equations are compatible with the Lagrangian field equations (3) only if suitable gauge conditions are chosen in order to eliminate the Lagrange multiplier u^3.

5. Gauge fixing and Dirac Brackets

At this stage we have a set of first-class constraints, the relations (13), that must be considered as generators of gauge transformations. The problem of choosing proper gauge conditions has to be solved to fully eliminate the redundant variables of the theory at the classical level and, therefore, to proceed with a consistent quantization of the Podolsky's field.

As it has already stated in the introduction section, the first attempt to find gauge conditions in the instant-form of the theory was made by using the Lorenz gauge $\partial_\mu A^\mu = 0$. However, as showed in [17], the Lorenz condition is not a good gauge choice for the Podolsky's field, since it does not fulfill the necessary requirements for a consistent gauge: it does not fix the gauge, it is not preserved by the equations of motion and it is not attainable. Moreover, it is also clear that the solutions of the field equations (3) cannot consist only by transverse fields.

The analysis of the correct gauge fixing on the null-plane can be made by closely inspect the EL equations of the system. If we look for the $\mu = +$ equation, it produces the explicit solution for A_+

$$A_+ = -\frac{1}{\left(1+a^2\Box\right)\nabla^2}\partial_+\left(1+a^2\Box\right)\left(\partial^- A_- + \partial^i A_i\right) \ , \tag{20}$$

where $\nabla^2 \equiv \partial_i\partial_i$. The remaining equations of motion can be written, eliminating the A_+ variable, by

$$\left(1+a^2\Box\right)\Box\mathbf{A}_- = 0 \ , \qquad \left(1+a^2\Box\right)\Box\mathbf{A}_i = 0 \ ,$$

with

$$\begin{aligned}
\mathbf{A}_- &\equiv A_- + \partial_-\frac{1}{\left(1+a^2\Box\right)\nabla^2}\left(1+a^2\Box\right)\left(\partial^- A_- + \partial^i A_i\right) \ , \\
\mathbf{A}_i &\equiv A_i + \partial_i\frac{1}{\left(1+a^2\Box\right)\nabla^2}\left(1+a^2\Box\right)\left(\partial^- A_- + \partial^j A_j\right) \ .
\end{aligned}$$

Therefore, we can achieve the variables $\mathbf{A}$ through a gauge transformation such that the gauge function is

$$\Delta = \frac{1}{\left(1+a^2\Box\right)\nabla^2}\left(1+a^2\Box\right)\left(\partial^- A_- + \partial^i A_i\right) \ .$$

In addition, these fields satisfy the condition

$$\left(1+a^2\Box\right)\left(\partial^-\mathbf{A}_- + \partial^i\mathbf{A}_i\right) = 0 \ , \tag{21}$$

which is the generalized Coulomb condition on the null-plane.

For that reason, the most natural gauge choice that is compatible with the field equations is given by

$$\left(1+a^2\Box\right)\left(\bar{A}_- + \partial^i A_i\right) \approx 0 \ . \tag{22}$$

Back to (20) we see that the time preservation of this relation is guaranteed if we set $A_+ \approx 0$. Whereas, consistency requires $\bar{A}_+ \approx 0$ as well.

In this gauge the field equations are written by

$$\left(1 + a^2\Box\right)\Box A_B = 0 \ ,$$

which is a generalized wave equation on the null-plane for the variables $A_B \equiv (A_-, A_i)$.

Back to the Hamiltonian framework, this analysis leads to the gauge conditions

$$\Omega_1 \equiv \bar{A}_+ \approx 0 \ , \tag{23a}$$

$$\Omega_2 \equiv A_+ \approx 0 \ , \tag{23b}$$

$$\Omega_3 \equiv \left(1 + a^2\Box\right)\left(\bar{A}_- + \partial^i A_i\right) \approx 0 \ , \tag{23c}$$

which is the generalized radiation gauge on the null-plane. The next step is to calculate Dirac Brackets for the set of ten constraints of the theory, but due to the present of the second-class constraints (14) it is more convenient to evaluate the reduced dynamics for these constraints first. Taking the matrix of the Poisson Brackets of the second-class constraints we have

$$M^{ij} \equiv 2\eta^{ij}\partial_-^x \begin{pmatrix} 0 & -2a^2\partial_-^x \\ 2a^2\partial_-^x & 1 - 2a^2\nabla_x^2 \end{pmatrix} \delta^3(x-y) \ . \tag{24}$$

The explicit evaluation of the inverse involves the knowledge of the inverse of the operators $\left(\partial_-^x\right)^{-1}$, $\left(\partial_-^x\right)^{-2}$, and $\left(\partial_-^x\right)^{-3}$, which are Green's functions of the operators ∂_-^x, $\left(\partial_-^x\right)^2$, and $\left(\partial_-^x\right)^3$. To achieve a unique solution it is necessary and sufficient to impose $\partial_-^x A_\mu = 0$, $\partial_-^x\partial_-^x A_\mu = 0$, and $\partial_-^x\partial_-^x\partial_-^x A_\mu = 0$ on $x^+ \to -\infty$ as the appropriate initial conditions of the theory. This choice is also consistent with the definition of momenta (6), since their definition are also dependent on initial-boundary conditions. Therefore, we write the unique inverse

$$N_{ij}\left(x,y\right) \equiv \frac{1}{2}\eta_{ij} \begin{pmatrix} \alpha\left(x,y\right) & \beta\left(x,y\right) \\ \gamma\left(x,y\right) & 0 \end{pmatrix} \tag{25}$$

with the coefficients

$$\alpha\left(x,y\right) = \frac{1}{4a^4}\left(x^- - y^-\right)^2 \epsilon\left(x^- - y^-\right)\left(1 - 2a^2\nabla_x^2\right)\delta^2(\mathbf{x}-\mathbf{y}) \ , \tag{26a}$$

$$\beta\left(x,y\right) = -\gamma(x,y) = \frac{1}{a^2}\ \left|x^- - y^-\right|\delta^2(\mathbf{x}-\mathbf{y}) \ . \tag{26b}$$

With this inverse we are able to define the first Dirac Brackets for two observables $A(x)$ and $B(y)$,

$$\{A(x), B(y)\}^* = \{A(x), B(y)\} - \iint d^3z d^3w \left\{A(x), \Phi_I^i(z)\right\} N_{ij}^{IJ}(z,w) \left\{\Phi_J^j(w), B(y)\right\} \ , \tag{27}$$

where $\{I, J\} = \{1, 2\}$. This definition implies elimination of the second-class constrains and the definition of an extended Hamiltonian where Φ_I are strongly zero. Thus, we are left with the first-class constraints Σ and the gauge conditions Ω. To proceed with the evaluation of the complete Dirac Brackets we should calculate the matrix of the first Dirac Brackets of these constraints. It is given by

$$C(x,y) \equiv \{\chi_A(x), \chi_B(y)\}^* = \begin{pmatrix} 0 & \mathcal{O}(x,y) \\ -\mathcal{O}^T(x,y) & 0 \end{pmatrix}$$

with $\chi_A \equiv (\Sigma_a, \Omega_a)$. If we write $\mathcal{D}_x \equiv \left(1 - a^2\nabla_x^2\right)$ the matrix $\mathcal{O}$ follows:

$$\mathcal{O}(x,y) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & \mathcal{D}_x\partial_-^x \\ 0 & 0 & -\mathcal{D}_x\nabla_x^2 \end{pmatrix} \delta^3(x-y) \ .$$

The inverse is given by

$$C^{-1}(x,y) = \begin{pmatrix} 0 & -\left(\mathcal{O}^{-1}\right)^T(x,y) \\ \mathcal{O}^{-1}(x,y) & 0 \end{pmatrix} \ , \tag{28}$$

in which

$$\mathcal{O}^{-1}(x,y)=\left(\begin{array}{ccc}-\delta^{3}(x-y) & 0 & 0\\ 0 & -\delta^{3}(x-y) & \gamma\left(x,y\right)\\ 0 & 0 & \rho\left(x,y\right)\end{array}\right)\ . \tag{29}$$

Under the considered boundary conditions the coefficients are given by

$$\gamma\left(x,y\right) = -\partial_{-}^{x}\left(\nabla_{x}^{2}\right)^{-1}\delta^{3}(x-y)\ , \tag{30a}$$

$$\rho\left(x,y\right) = -\mathcal{D}_{x}^{-1}\left(\nabla_{x}^{2}\right)^{-1}\delta^{3}(x-y)\ , \tag{30b}$$

where

$$\left(\nabla_{x}^{2}\right)^{-1}\delta^{3}(x-y)=\frac{1}{4\pi}\ln\left(\mathbf{x}-\mathbf{y}\right)^{2}.$$

Then, we are able to define the complete Dirac Brackets of the generalized radiation gauge:

$$\begin{aligned}\left\{A(x),B(y)\right\}^{**} \equiv\ & \left\{A(x),B(y)\right\}^{*}+\int d^{3}zd^{3}w\left\{A(x),\Sigma_{a}(z)\right\}^{*}\left[\left(\mathcal{O}^{-1}\right)^{T}\right]^{ab}(z,w)\left\{\Omega_{b}(w),B(y)\right\}^{*}\\ & -\int d^{3}zd^{3}w\left\{A(x),\Omega_{a}(z)\right\}^{*}\left[\mathcal{O}^{-1}\right]^{ab}(z,w)\left\{\Sigma_{b}(w),B(y)\right\}^{*}\ .\end{aligned} \tag{31}$$

The complete set of fundamental DB follows:

$$\begin{aligned}\left\{A_{\mu}(x),\bar{A}_{\nu}(y)\right\}^{**} &= -\frac{1}{8a^{2}}\delta_{\mu}^{i}\delta_{\nu}^{j}\eta_{ij}\left|x^{-}-y^{-}\right|\delta^{2}(\mathbf{x}-\mathbf{y}),\\ \left\{A_{\mu}(x),p^{\nu}(y)\right\}^{**} &= \left[\delta_{\mu}^{\nu}-\delta_{\mu}^{+}\delta_{+}^{\nu}+\left(\delta_{\mu}^{-}\partial_{-}+\delta_{\mu}^{i}\partial_{i}\right)\left(\delta_{+}^{\nu}\partial_{-}+\delta_{j}^{\nu}\partial^{j}\right)\frac{1}{\nabla^{2}}\right]\delta^{3}(x-y),\\ \left\{A_{\mu}(x),\pi^{\nu}(y)\right\}^{**} &= \delta_{-}^{\nu}\left[\delta_{\mu}^{-}\partial_{-}+\delta_{\mu}^{i}\partial_{i}\right]\frac{1}{\nabla^{2}}\delta^{3}(x-y),\\ \left\{\bar{A}_{\mu}(x),\bar{A}_{\nu}(y)\right\}^{**} &= \frac{1}{64a^{4}}\delta_{\mu}^{i}\delta_{\nu}^{j}\eta_{ij}\left(x^{-}-y^{-}\right)^{2}\epsilon\left(x^{-}-y^{-}\right)\left[1-2a^{2}\nabla^{2}\right]\delta^{2}(\mathbf{x}-\mathbf{y})\\ &\quad -\frac{1}{8a^{2}}\eta_{ij}\left|x^{-}-y^{-}\right|\left[\delta_{\mu}^{i}\delta_{\nu}^{-}+\delta_{\mu}^{-}\delta_{\nu}^{i}\right]\partial_{j}\delta^{2}(\mathbf{x}-\mathbf{y}),\\ \left\{\bar{A}_{\mu}(x),p^{\nu}(y)\right\}^{**} &= \frac{1}{8a^{2}}\delta_{\mu}^{i}\delta_{-}^{\nu}\left|x^{-}-y^{-}\right|\partial_{i}\delta^{2}(\mathbf{x}-\mathbf{y})\\ &\quad -\frac{1}{4}\delta_{\mu}^{i}\epsilon\left(x^{-}-y^{-}\right)\left[\delta_{k}^{\nu}\partial_{k}\partial_{i}+\frac{1}{2a^{2}}\delta_{i}^{\nu}\left(1-2a^{2}\nabla^{2}\right)\right]\delta^{2}(\mathbf{x}-\mathbf{y})\\ &\quad -\delta_{+}^{\nu}\left[\left[\delta_{\mu}^{-}\partial_{-}+\delta_{\mu}^{i}\partial_{i}\right]-\frac{1}{2}\delta_{\mu}^{i}\partial_{i}\right]\delta^{3}(x-y),\\ \left\{\bar{A}_{\mu}(x),\pi^{\nu}(y)\right\}^{**} &= \left[\delta_{\mu}^{\nu}-\delta_{\mu}^{+}\delta_{+}^{\nu}+\eta_{\mu j}\delta_{j}^{\nu}\right]\delta^{3}(x-y)-\frac{1}{4}\eta_{\mu j}\delta_{-}^{\nu}\epsilon\left(x^{-}-y^{-}\right)\partial_{j}\delta^{2}(\mathbf{x}-\mathbf{y}).\end{aligned}$$

With these brackets we can deduce the fundamental ones that will lead, through the correspondence principle, to a consistent quantization of the field. The physical degrees of freedom can be found with the analysis of the constraints as strong relations. Of course, the fields A_{+}, $\bar{A}_{+}$, π^{+} and π^{i} are not independent, since they are strongly zero in the formalism. Thus, p^{i}, p^{+} and p^{-} can be written in function of π^{-} and other variables. The gauge condition (23c) also eliminates $\bar{A}_{-}$. Therefore, the only independent variables are actually given by $A_{-},A_{i},\bar{A}_{i},p^{i}$ and π^{-}. They are eight independent fields, less than the dynamics in instant-form [17], which can be seen as a good feature, but the structure of the phase space comes out to be quite more complicate.

Considering, for example, the brackets

$$\left\{A_{-}(x),p^{i}(y)\right\}^{**}=\frac{\partial_{-}\partial^{i}}{\nabla^{2}}\delta^{3}(x-y), \tag{32}$$

we can see that the longitudinal field acquires a non-local character. This is expected for every system analyzed on the null-plane, since this component lies on the light-cone and no criterium of causality can be employed for this field. The non-locality is due to the second-class constraints, which does not appear in instant-form dynamics of the system.

For the transverse fields the brackets

$$\left\{A_i(x), p^j(y)\right\}^{**} = \left[\delta_i^j - \frac{\partial_i\partial_j}{\nabla^2}\right]\delta^3(x-y) \tag{33}$$

indicates that a Coulomb-type interaction is present, this case in two dimensions, what justifies to call the gauge condition (22) the generalized Coulomb condition. This is also expected, since these brackets depends exclusively on the first-class constraints plus gauge conditions, just like in the instant-form.

6. Final remarks

We have analysed the canonical structure of Podolsky's electrodynamics on the null-plane. The theory has high-order derivatives in the Lagrangian function, so we followed the procedure outlined in [17] for the definition of the Hamiltonian density (5) and of the canonical momenta associated to the fields A_μ and $\bar{A}_\mu$ (6), which result from the definition of the conserved energy-momentum tensor.

We have observed in the study of the initial-boundary problem of Podolsky's equation that, because it is a second-order equation, the uniqueness of the solution is obtained when the field A_μ is specified on the null-plane $x^+ = cte$ and three boundary conditions are imposed on $x^- = cte$. These conditions where chosen to be $\partial_- A_\mu = 0$, $(\partial_-)^2 A_\mu = 0$, and $(\partial_-)^3 A_\mu = 0$ on $x^- \to -\infty$.

In the canonical analysis of the Podolsky's theory we found a set of three first-class constraints (13), and a set of four second-class ones (14). The first-class constraints are responsible for the $U(1)$ invariance of the Action, which is expected since the gauge character of the field should not be destroyed by the choice of parametrization. The form of this set is analogous to the set found in instant-form [17], which is also expected.

The new feature on the null-plane is the second-class constraints, which are not present in the conventional instant-form dynamics [17]. The appearance of second-class constraints is a common effect of the null-plane dynamics [30,31,32,33], and they are responsible to the fact that the analysis on the null-plane requires a lesser number of degrees of freedom. Because of the second-class constraints the longitudinal components of the fields turned out to be non-local.

To evaluate the physical degrees of freedom it was necessary to choose proper gauge conditions for the theory, which was a subject that needed closer inspection. Gauge conditions must obey a set of requirements to be consistent with the formalism: they must fix completely the gauge, they must be consistent with the field equations, they must not affect Lorentz covariance and, last but not least, they must be attainable. Therefore, we followed the procedure outlined in [17] and found that the generalized radiation gauge (23) on the null-plane fulfill all these requirements. Of course, this gauge choice is not the only consistent possible choice. There is, for example, the so called null-plane gauge, which will be studied in a future work concerning the Podolsky's field coupled with scalar and spinor fields.

Since the first and second-class constraints, together with the gauge conditions were known, we calculated the Dirac Brackets that had clarified the physical fields of the system. However, these brackets are not unique unless we specify all the information about the initial-boundary value problem of the theory. By imposing the value of the field on the null-plane $x^+ = cte$, and the considered boundary conditions on $x^- = cte$, we have fixed the hidden subset of the first-class constraints [30,29] and got a unique inverse for the second-class constraints matrix when the ambiguity on the operators $\left(\partial_-^x\right)^{-1}$, $\left(\partial_-^x\right)^{-2}$, and $\left(\partial_-^x\right)^{-3}$ was eliminated.

Finally, an analysis of the physical fields results in the true degrees of freedom, which are given by $A_-, A_i, \bar{A}_i, p^i$ and π^-. The complete Dirac Brackets of these fields implicated the non-locality of the longitudinal component A_- and a Coulomb-type interaction in the electrostatic case, in two dimensions.

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