QED_2 on the null plane using Faddeev-Jackiw quantization

B.M. Pimentel

Instituto de Física Teórica (IFT/UNESP), UNESP - São Paulo State University,São Paulo, SP, Brazil

G. R. Zambrano

Departamento de Física, Universidad de Nariño, San Juan de Pasto, Nariño, Colombia

Abstract

Half a century ago Dirac has proposed three different forms of relativistic dynamics depending on the types of surfaces where independent modes were initiated. The first possibility when a space-like surface is chosen (*instant form*) has been used most frequently so far and is usually called equal-time quantization. The second choice is to the a surface of a single light wave (front form or null-plane). The third possibility is to take a branch of hyperbolic surface (point form). In this paper we are going to study QED_2 on the null-plane and we will show that one of the first class constraints of the theory has a contribution provided by the scalar sector and in addition the theory have a second class constraint in the scalar sector which is manifest in the free case. It is not natural in the instant form. The Faddeev-Jackiw procedure for constrained system is applied to calculate the commutation relations of the theory.

Introduction

In 1949 Dirac pointed out that in a relativistic quantum theory the choice of the time variable is not unique [1]. In his "front form" (or null-plane as we call) of dynamics he suggested the use light front $x^+ \equiv x^0 + x^3$ as surfaces of equal time. The initial conditions are give in this hyperplane and correspondingly, the commutation relations are prescribed on a plane $x^+ = cte$. Already in the early seventies it was noted that quantizing on the null-plane means quantizing on the characteristic surface of the classical field equations [2]. The conclusion was that one is dealing with a characteristic value problem when one wants to solve these equations .

A general characteristic of a relativistic theory on null-plane is what it describes a dynamical systems with constraints, thus, the Faddeev-Jackiw procedure for constrained system [3] is used to quantize the theory. In this work we are going to derive the commutation relations among the fundamental dynamical variables of the theory.

QED_2 on the null-plane coordinates

The $U(1)$ gauge theory we are considering is defined by the following Lagrangian density in 2-dimensional space-time

$$\begin{aligned}\mathcal{L}^{(0)} = & -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{i}{2}\bar{\varphi}\gamma^+\partial_+\varphi - \frac{i}{2}\partial_+\bar{\varphi}\gamma^+\varphi + \frac{i}{2}\bar{\varphi}\gamma^-\partial_-\varphi - \frac{i}{2}\partial_-\bar{\varphi}\gamma^-\varphi \\ & - m\bar{\varphi}\varphi - gA_\mu\bar{\varphi}\gamma^\mu\varphi. \end{aligned} \tag{1}$$

From (1) it is easy to write the first-order Lagrangian by introducing the canonical momentum $(\Pi^\nu, \bar{\pi}, \pi)$ with respect to the fields $(A_\nu, \varphi, \bar{\varphi})$ A_μ and θ, respectively. The initial set of symplectic variables defining the extended space is given by the set $\xi_k^{(0)} = (\varphi_a, \bar{\varphi}_a, A_+, A_-, \Pi^-)$, and so the starting Lagrangian density is written in first order as follow:

$$\mathcal{L}^{(0)} = \Pi^-\partial_+A_- + \frac{i}{2}\bar{\varphi}\gamma^+\partial_+\varphi - \frac{i}{2}\partial_+\bar{\varphi}\gamma^+\varphi - \mathcal{H}^{(0)}, \tag{2}$$

where the zero iterated symplectic potential has the following form:

$$\mathcal{H}^{(0)} = \frac{1}{2}\Pi^-\Pi^- + \Pi^-\partial_-A_+ - \frac{i}{2}\bar{\varphi}\gamma^-\partial_-\varphi + \frac{i}{2}\partial_-\bar{\varphi}\gamma^-\varphi + m\bar{\varphi}\varphi + gA_\mu\bar{\varphi}\gamma^\mu\varphi. \tag{3}$$

Classically the fields are described by Grassmann variables [5], thus, the zero iterated symplectic two-form matrix is defined by [6]:

$$M_{AB}^{(0)}(\mathbf{x},\mathbf{y}) = \frac{\delta K_B^{(0)}(\mathbf{y})}{\delta\xi_A^{(0)}(\mathbf{x})} - (-1)^{n_A n_B}\frac{\delta K_A^{(0)}(\mathbf{x})}{\delta\xi_B^{(0)}(\mathbf{y})}. \tag{4}$$

where n_A specify the parity of the field $\xi_A^{(0)}(\mathbf{x})$. From (4) it is possible determine the following the components:

$$M_{AB}^{(0)}(\mathbf{x},\mathbf{y}) = -i\begin{pmatrix} & \varphi_b(\mathbf{y}) & \bar{\varphi}_b(\mathbf{y}) & A_+(\mathbf{y}) & A_-(\mathbf{y}) & \Pi^-(\mathbf{y}) \\ \varphi_a(\mathbf{x}) & 0 & (\gamma^+)_{ab}^T & 0 & 0 & 0 \\ \bar{\varphi}_a(\mathbf{x}) & \gamma_{ab}^+ & 0 & 0 & 0 & 0 \\ A_+(\mathbf{x}) & 0 & 0 & 0 & 0 & 0 \\ A_-(\mathbf{x}) & 0 & 0 & 0 & 0 & -1 \\ \Pi^-(\mathbf{x}) & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\delta^3(\mathbf{x}-\mathbf{y}). \tag{5}$$

In the null-plane coordinates the γ^+ matrix is singular, then $M^{(0)}(\mathbf{x},\mathbf{y})$ is singular too. It is possible to show that the matrix (5) has three zero modes associated with the following set of Lagrangian constraints:

$$\begin{aligned} \Omega_1^{(0)} &= \bar{\varphi}(\mathbf{x})\left[\gamma^-\left(i\partial_-^x + gA_-(\mathbf{x})\right) + m\right]\Delta^- = 0, \\ \Omega_2^{(0)} &= \Delta^+\left[\gamma^-\left(i\partial_-^x - gA_-(\mathbf{x})\right) - m\right]\varphi(\mathbf{x}) = 0. \\ \Omega_3^{(0)} &= \partial_-^x\Pi^-(\mathbf{x}) - g\bar{\varphi}(\mathbf{x})\gamma^+\varphi(\mathbf{x}) = 0. \end{aligned} \tag{6}$$

where the projectors Δ^+ and Δ^- [7] have been used. According to the symplectic algorithm, the constraint (6) are introduced in the Lagrangian density by using Lagrangian multipliers, thus, the first iterated Lagrangian density is written as:

$$\begin{aligned}\mathcal{L}^{(1)} = & \Pi^-\partial_+A_- - \frac{i}{2}(\partial_+\varphi_c)\bar{\varphi}_b\gamma_{bc}^+ - \frac{i}{2}(\partial_+\bar{\varphi}_b)\gamma_{bc}^+\varphi_c - \dot{\lambda}_b\Omega_{1_b}^{(0)} \\ & + \dot{\bar{\lambda}}_b\Omega_{2_b}^{(0)} + \Omega_3^{(0)}\dot{\beta} - \mathcal{H}^{(1)}\end{aligned} \tag{7}$$

where the first iterated symplectic potential is

$$\mathcal{H}^{(1)} = \mathcal{H}^{(0)}_{\Omega_i^{(0)}=0} = \frac{1}{2}\Pi^-\Pi^- - \frac{i}{2}\bar{\varphi}\gamma^-\partial_-\varphi + \frac{i}{2}\partial_-\bar{\varphi}\gamma^-\varphi + m\bar{\varphi}\varphi + gA_-\bar{\varphi}\gamma^-\varphi$$

Now, we enlarged the space with the first iterated set of symplectic variables defined by $\xi_k^{(1)} = (\varphi_a, \bar{\varphi}_a, A_-, \Pi^-, \lambda, \bar{\lambda}, \beta)$. The first iterated symplectic matrix is written as:

$$M_{AB}^{(1)}(\mathbf{x},\mathbf{y}) = \frac{\delta K_B^{(1)}(\mathbf{y})}{\delta\xi_A^{(1)}(\mathbf{x})} - (-1)^{n_A n_B}\frac{\delta K_A^{(1)}(\mathbf{x})}{\delta\xi_B^{(1)}(\mathbf{y})}.$$

which can be as:

$$M_{AB}^{(1)}(\mathbf{x},\mathbf{y}) = \left(\begin{array}{cccc} 0 & -i(\gamma^+)_{ab}^T & 0 & 0 \\ -i\gamma_{ab}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & \left\{\left[\gamma^-\left(i\partial_-^x + gA_-(\mathbf{x})\right) + m\right]\Delta^-\right\}_{ab}^T & -g\left[\bar{\varphi}(\mathbf{x})\gamma^-\Delta^-\right]_a & 0 \\ \left\{\Delta^+\left[\gamma^-\left(i\partial_-^x - gA_-(\mathbf{x})\right) - m\right]\right\}_{ab} & 0 & g\left[\Delta^+\gamma^-\varphi(\mathbf{x})\right]_a & 0 \\ -g\left[\bar{\varphi}(\mathbf{x})\gamma^+\right]_b & g\left[\gamma^+\varphi(\mathbf{x})\right]_b & 0 & -\partial_-^x \end{array}\right.$$
$$\left.\begin{array}{ccc} 0 & -\left\{\Delta^+\left[\gamma^-\left(i\partial_-^x + gA_-(\mathbf{x})\right) + m\right]\right\}_{ab}^T & g\left[\bar{\varphi}(\mathbf{x})\gamma^+\right]_a \\ -\left\{\left[\gamma^-\left(i\partial_-^x - gA_-(\mathbf{x})\right) - m\right]\Delta^-\right\}_{ab} & 0 & -g\left[\gamma^+\varphi(\mathbf{x})\right]_a \\ g\left[\bar{\varphi}(\mathbf{x})\gamma^-\Delta^-\right]_b & -g\left[\Delta^+\gamma^-\varphi(\mathbf{x})\right]_b & 0 \\ 0 & 0 & -\partial_-^x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)\delta^3(\mathbf{x}-\mathbf{y})$$

: It is possible to show that the matriz is singular, however, the zero mode will not give rise to a new Lagrangian constraint, then the symmetric matrix continues being singular what characterizes that quantum electrodynamics in null plane coordinates it is still a gauge theory. In order to obtain a regular symplectic matrix a gauge fixing term must be added to the symplectic potential. We choose the Coulomb gauge $\Theta = A_-(\mathbf{x}) = 0$. Using the consistency condition by Lagrange multiplier $\eta(\mathbf{x})$, which will increase the size of the configuration space, we obtain the second iterative Lagrangian, i.e.:

$$\begin{aligned}\mathcal{L}^{(2)} = & \Pi^-\partial_+A_- - \frac{i}{2}(\partial_+\varphi_c)\bar{\varphi}_b\gamma_{bc}^+ - \frac{i}{2}(\partial_+\bar{\varphi}_b)\gamma_{bc}^+\varphi_c + \Omega_1^{(0)}\dot{\lambda} + \dot{\bar{\lambda}}\Omega_2^{(0)} \\ & + \Omega_3^{(0)}\dot{\beta} + \Theta\dot{\eta} - \mathcal{H}^{(2)}\end{aligned} \tag{8}$$

where

$$\mathcal{H}^{(2)} = \mathcal{H}^{(1)}_{\Omega_i^{(0)},\Omega^{(1)},\Theta=0} = \frac{1}{2}\Pi^-\Pi^- - \frac{i}{2}\bar{\varphi}\gamma^-\partial_-\varphi + \frac{i}{2}\partial_-\bar{\varphi}\gamma^-\varphi + m\bar{\varphi}\varphi. \tag{9}$$

As before, we set the symplectic variable $\xi_k^{(2)} = (\varphi_a, \bar{\varphi}_a, A_-, \Pi^-, \lambda, \bar{\lambda}, \beta, \eta)$. Now, from (8) we obtain the second-iterated symplectic two-form matrix,

$$M_{AB}^{(2)}(\mathbf{x},\mathbf{y}) = \frac{\delta K_B^{(2)}(\mathbf{y})}{\delta\xi_A^{(2)}(\mathbf{x})} - (-1)^{n_A n_B}\frac{\delta K_A^{(2)}(\mathbf{x})}{\delta\xi_B^{(2)}(\mathbf{y})}.$$

Since this matrix is not singular, we finally have the inverse matrix after a laborious calculation. Then, we determine the following generalized bracket:

$$\{\varphi_a(\mathbf{x}), \bar{\varphi}_b(\mathbf{y})\} = -\frac{im^2}{8}|x^- - y^-|\gamma_{ab}^+ + \frac{m}{4}\epsilon(x^- - y^-)\mathbf{I}_{ab} + \frac{i}{2}\delta(x^- - y^-)\gamma_{ab}^-, \tag{10}$$

where $\mathbf{I}$ is the identity matrix. IN the similar way is possible determine:

$$\begin{aligned}\left\{\varphi_1(\mathbf{x}), \Pi^-(\mathbf{y})\right\} &= -\frac{ig}{2}\varphi_1(\mathbf{x})\epsilon(x^- - y^-) \\ \left\{\varphi_2(\mathbf{x}), \Pi^-(\mathbf{y})\right\} &= -\frac{mg}{4\sqrt{2}}\int dv\epsilon(x^- - v^-)\varphi_1(\mathbf{v})\epsilon(v^- - y^-)\end{aligned} \tag{11}$$

The relations (10) and (11) are consistent with the brackets derived with the Dirac method to the QED_2 on the null plane coordinates in the null-plane gauge condition.

Conclusions

In this paper we have studies QED_2 with the symplectic quantization method. The results give us the Dirac brackets of the theory, which is an alternative to the orthodox Dirac method on constrained dynamics [**?**]. At the same time, we have shown that the symplectic approach is more intuitive in the sense that the constraints are related to the generalized canonical momenta and the Lagrange multipliers to the symplectic variables in the enlarged symplectic structure of the constrained manifold. For the QED_2 we have shown that the number of the constraints is fewer and the structure of these constraints is very simple because we do not need to distinguish first or second class constraints, primary or secondary constraints, etc. We have easily obtained the generalized brackets by reading directly from the inverse matrix $\left[f^{AB(2)}\right]^{-1}$ of the symplectic two form matrix. Finally, we can observe that the potential symplectic obtained at the final stage of iterations is exactly the Hamiltonian which is obtained through several steps with the usual Dirac formulation of the constrained systems.

References

[1] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[2] R. A. Neville and F. Rohrlich, Nuovo Cimento, **A1** , 625 (1971).

[3] L. Faddeev and R. Jackiw, Phys. Rev. Lett., **60**, 1692 (1988).

[4] R. A. Neville and F. Rohrch, Phys. Rev. **D3**,692(1971).

[5] R. Casalbuoni , Nuouo Cimento **A33**, 115, 389 (1976).

[6] J. Govaerts, Int. J. Mod. Phys. **A5**, 3625 (1990).
E. C. Manavella, Int. J. Mod. Phys. **A27**, 1250145 (2012).

[7] R. Casana , B.M. Pimentel and G.E.R. Zambrano, Int. J. Mod. Phys. **E16**, 2993 (2007).