Non Abelian $SQED_4$ in the null-plane gauge

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Abstract

We are going to study the non-abelian scalar electrodynamics ($SQED_4$) on the null-plane formalism. We follow the Dirac's technique for constrained systems to perform a detailed analysis of the constraint structure and late we give the generalized Dirac brackets for the physical variables.

Introduction

To quantize the theory on the null-plane, initial conditions on the hyperplane $x^+ = cte$ and equal x^+ commutation relations must be given and the hamiltonian describe the time evolution from an initial value surface to other parallel surface that intersect the x^+ axis at some later time. Although the prescription has a lot of similarities with the conventional approach, there are significant difference in the quantization. The null-plane framework is a constrained dynamical theory where second class constraints arise. These can be eliminated by constructing Dirac brackets (DB) and the theory can be quantized canonically by the correspondence principle in terms of a reduced number of independent fields. Thus, if the Dirac's method [1] is employed, it allows identify the independent fields and the null plane hamiltonian and the commutation relations will be constructed in terms of them.

The quantization of relativistic field theory at the null plane time, which was proposed by Dirac [2], has found important applications [3] in both gauge theories and string theory [4]. It has been conventional to apply the null-plane quantization to gauge theory in the null-plane gauge A_-, since the transverse degrees of freedom on the gauge field can be immediately identified as the dynamical degrees of freedom, and ghost fields can be ignored in the quantum action of the non-abelian gauge theory [5].

In this paper we will discuss the null-plane quantization of the Yang-Mills gauge field and the Non-abelian scalar electrodynamics following the Dirac's formalism to constrained systems.

NON-ABELIAN $SQED_4$

In the adjoint representation, the theory is describe by the following lagrangian density

$$\mathcal{L} = \eta^{\mu\nu} \left(D_\mu\right)^{ab} \Phi_b^\dagger \left(D_\nu\right)^{ac} \Phi_c - m^2 \Phi_a^\dagger \Phi_a - \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a, \qquad (1)$$

where the field strength $F_{\mu\nu}^a$ and the covariant derivative $\left(D_\mu\right)^{ab}$ are defined, respectively, by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\varepsilon_{abc} A_\mu^b A_\nu^c \quad , \quad \left(D_\mu\right)^{ab} \equiv \delta_a^b \partial_\mu^x - g\varepsilon_{abc} A_\mu^c \qquad (2)$$

and where Φ_c is a field scalar which has three components in an "internal" space and the gauge transformation are rotations in this space. This will give a "vector" quantity which is conserved, it will be like isospin.

The field equations are given for

$$\begin{aligned} \left(D_\nu\right)^{ab} F_b^{\nu\mu} &= J_a^\mu \\ \left[\eta^{\alpha\beta} \left(\delta_a^b \partial_\alpha - g\varepsilon_{abf} A_\alpha^f\right) \left(\delta_b^c \partial_\beta - g\varepsilon_{bcg} A_\beta^g\right) + \delta_a^c m^2\right] \Phi_c &= 0 \\ \Phi_c^\dagger \left[\eta^{\alpha\beta} \left(\delta_c^b \partial_\alpha + g\varepsilon_{cbf} A_\alpha^f\right) \left(\delta_b^a \partial_\beta + g\varepsilon_{bag} A_\beta^g\right) + \delta_a^c m^2\right] &= 0, \end{aligned} \qquad (3)$$

where J_h^β is the current density defined by

$$J_h^\beta \equiv g\eta^{\beta\alpha} \varepsilon_{hab} \left\{ \left[\left(D_\alpha\right)^{ac} \Phi_c^\dagger \right] \Phi_b + \Phi_b^\dagger \left[\left(D_\alpha\right)^{ac} \Phi_c \right] \right\} \qquad (4)$$

The canonical conjugate momenta to the fields $\left(A_\mu^a,\ \Phi_a,\ \Phi_a^\dagger\right)$ are

$$\pi_a^\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_+ A_\mu^a\right)} = -F_a^{+\mu} \quad , \quad \Pi_a^\dagger \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_+ \Phi_a\right)} = \left(D_-^x\right)^{ab} \Phi_b^\dagger \quad , \quad \Pi_a \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_+ \Phi_a^\dagger\right)} = \left(D_-^x\right)^{ab} \Phi_b \qquad (5)$$

respectively. From (5) we get the following set of primary constraints:

$$\begin{aligned} \phi_a \equiv \pi_a^+ \approx 0 \quad &, \quad \phi_a^k \equiv \pi_a^k - \partial_- A_k^a + \partial_k A_-^a - g\varepsilon_{abc} A_-^b A_k^c \approx 0, \\ \Theta_a \equiv \Pi_a - \left(D_-^x\right)^{ab} \Phi_b \approx 0 \quad &, \quad \Theta_a^\dagger \equiv \Pi_a^\dagger - \left(D_-^x\right)^{ab} \Phi_b^\dagger \approx 0. \end{aligned} \qquad (6)$$

The primary hamiltonian of the theory is

$$\begin{aligned} H_P = \int d^3y \Big\{ & \frac{1}{2}\left(\pi_b^-\right)^2 + \left[\pi_b^- \left(D_-^y\right)^{bc} + \pi_b^i \left(D_i^y\right)^{bc} + J_c^+\right] A_+^c + \left[\left(D_i^x\right)^{ab} \Phi_b^\dagger\right] \left[\left(D_i^x\right)^{ad} \Phi_d\right] \\ & + m^2 \Phi_b \Phi_b + \frac{1}{4}\left(F_{ij}^b\right)^2 + u^b \phi_b + \lambda_l^b \phi_b^l + U_b^\dagger \Theta_b + \Theta_b^\dagger U_b \Big\}, \end{aligned} \qquad (7)$$

where $\left(u^b,\ \lambda_l^b\right)$ are the lagrange multipliers associated to the vector constraints and $\left(U_b^\dagger,\ U_b\right)$ are the multipliers associated with the scalar ones.

Following the Dirac's procedure, we determine the consistence conditions on the primary constraints, thus, such requirement on the scalar constraints yield:

$$\begin{aligned} \dot{\Theta}_d &= -g\varepsilon_{dbf} \Phi_f \pi_b^- - 2g\varepsilon_{bcf} \left(D_-^x\right)^{db} \left[\Phi_f A_+^c\right] + \left(D_i^x\right)^{da} \left(D_i^x\right)^{ae} \Phi_e - m^2 \Phi_d - 2\left(D_-^x\right)^{db} U_b \\ \dot{\Theta}_d^\dagger &= -g\varepsilon_{dbf} \Phi_f^\dagger \pi_b^- - 2g\varepsilon_{bcf} \left(D_-^x\right)^{db} \left[\Phi_f^\dagger A_+^c\right] + \left(D_i^x\right)^{da} \left(D_i^x\right)^{ab} \Phi_b^\dagger - m^2 \Phi_d^\dagger - 2\left(D_-^x\right)^{db} U_b^\dagger. \end{aligned} \qquad (8)$$

This relations determine conditions on the multipliers $\left(U_b^\dagger,\ U_b\right)$, respectively, and there are not more constraints associated with the scalar sector. In the vector sector, the consistence conditions on ϕ_a^k give

$$\dot{\phi}_a^k = \left(D_k^x\right)^{ab} \pi_b^- + \left(D_i^x\right)^{ab} F_{ik}^b - J_a^k - 2\left(D_-^x\right)^{ab} \lambda_k^b \approx 0 \qquad (9)$$

an equation for its associated multiplier λ_k^b. Finally, the consistence condition on π_a^+ give a secondary constraint

$$\dot{\phi}_a = \left(D_-^x\right)^{ab} \pi_b^- + \left(D_i^x\right)^{ab} \pi_b^i - J_a^+ \equiv G_a \approx 0, \qquad (10)$$

which it is the Gauss' law. It is possible, after a laborious work, to verify that not more further constraints are generated from the consistence condition of the Gauss' law because it is automatically conserved

$$\dot{G}_a = g\varepsilon_{adf} \left[\Phi_f^\dagger \dot{\Theta}_d + \Phi_f \dot{\Theta}_d^\dagger\right] \approx 0. \qquad (11)$$

Then, the constraints (6), (**??**) and (10) constitute the full set of constraints of the theory.

CONSTRAINTS CLASSIFICATION

The non null PB's among the constraints of the theory are

$$\begin{aligned} &\left\{\Theta_a^\dagger(x), \Theta_b(y)\right\} = -2\left(D_-^x\right)^{ab} \delta^3(x-y) \qquad &&\left\{\phi_a^k(x), \phi_b^l(y)\right\} = -2\delta_k^l \left(D_-^x\right)^{ab} \delta^3(x-y) \\ &\left\{G_a(x), \Theta_b^\dagger(y)\right\} = -2g\varepsilon_{acf} \Phi_f^\dagger(x) \left(D_-^x\right)^{cb} \delta^3(x-y) \qquad &&\left\{G_a(x), \Theta_b(y)\right\} = -2g\varepsilon_{acf} \Phi_f(x) \left(D_-^x\right)^{cb} \delta^3(x-y), \end{aligned} \qquad (12)$$

thus, we can observe that π_a^+ has vanishing PB with all the other constraints, therefore, it is a first class constraint. The remaining set, $\left(\phi_a^k,\ \Theta_a,\ \Theta_a^\dagger,\ G_a\right)$, is apparently a second class set, however, it is possible to show that the matrix formed with these set of constraints is singular, it is because the matrix has a zero mode whose eigenvector gives a linear combination of constraints which is one first class constraint[6], such eigenvector is

$$\Sigma_a \equiv G_a - g\varepsilon_{adf} \left[\Phi_f^\dagger \Theta_d + \Phi_f \Theta_d^\dagger\right]. \qquad (13)$$

Then, the theory has the following set of second class constraints

$$\phi_a^k \equiv \pi_a^k - \partial_-^x A_k^a + \partial_k^x A_-^a - g\varepsilon_{afg} A_-^f A_k^g \quad , \quad \Theta_a = \Pi_a - \left(D_-^x\right)^{ab} \Phi_b \quad , \quad \Theta_a^\dagger = \Pi_a^\dagger - \left(D_-^x\right)^{ab} \Phi_b^\dagger, \qquad (14)$$

and the set of first class constraints

$$\phi_a \equiv \pi_a^+ \quad , \quad \Sigma_a \equiv G_a - g\varepsilon_{adf} \left[\Phi_f^\dagger \Theta_d + \Phi_f \Theta_d^\dagger\right]. \qquad (15)$$

We can sure that (15) is the maximal number of first class constraints of the theory, since, the consistence condition on $\left(\phi_a^k,\ \Theta_a,\ \Theta_a^\dagger\right)$ lead to equations that determine their respective lagrange multipliers.

DIRAC BRACKETS

We have the following set of second class constraints:

$$\begin{aligned} &\Sigma_1 \equiv \pi_a^+ \quad , \quad \Sigma_2 \equiv G_a - g\varepsilon_{adf} \left[\Phi_f^\dagger \Theta_d + \Phi_f \Theta_d^\dagger\right] \quad , \quad \Sigma_3 \equiv A_-^a \\ &\Sigma_4 \equiv \pi_a^- + \partial_-^x A_+^a \quad , \quad \Sigma_5 \equiv \pi_a^k - \partial_-^x A_k^a + \partial_k^x A_-^a - g\varepsilon_{afg} A_-^f A_k^g \\ &\Sigma_6 \equiv \Pi_a - \left(D_-^x\right)^{ab} \Phi_b \quad , \quad \Sigma_7 \equiv \Pi_a^\dagger - \left(D_-^x\right)^{ab} \Phi_b^\dagger, \end{aligned} \qquad (16)$$

with these, we define the following constraint matrix $D_{ab}(x,y) \equiv \left\{\Sigma_a(x), \Sigma_b(y)\right\}$, from where the DB for the dynamical variables are determined if a explicit evaluation of the inverse of this matrix is does. If we consider the appropriate boundary conditions on the fields,[6][**?**], a unique inverse of the constraint matrix is obtained and after a laborious work we obtain, the aim of this paper, the DB for the independent dynamical variables of non-abelian $SQED_4$.

$$\begin{aligned} \left\{A_+^a(x), A_+^b(y)\right\}_D &= -\frac{1}{8} \left(D_s^x\right)^{ac} \left(D_s^x\right)^{cb} \delta^2\left(x^\intercal - y^\intercal\right) \int du\, \epsilon(x-u)\, \epsilon\, |u-y| \\ \left\{A_k^a(x), A_l^b(y)\right\}_D &= -\frac{1}{4} \delta_b^a \delta_k^l \epsilon(x-y)\, \delta^2\left(x^\intercal - y^\intercal\right) \\ \left\{A_k^a(x), A_+^b(y)\right\}_D &= \frac{1}{4} |x-y| \left(D_k^x\right)^{ab} \delta^2\left(x^\intercal - y^\intercal\right) \\ \left\{\Phi_a(x), \Phi_b^\dagger(y)\right\}_D &= -\frac{1}{4} \delta_a^b \epsilon(x-y)\, \delta^2\left(x^\intercal - y^\intercal\right) \\ \left\{\Phi_a(x), A_+^b(y)\right\}_D &= \frac{g}{2} \varepsilon_{abf} \delta^2\left(x^\intercal - y^\intercal\right) \left\{\Phi_f(x)\, |x-y| - \frac{1}{4} \int dv\, \Phi_f(v)\, \epsilon(x-v)\, \epsilon(v-y)\right\} \\ \left\{\Phi_a^\dagger(x), A_+^b(y)\right\}_D &= \frac{g}{2} \varepsilon_{abf} \delta^2\left(x^\intercal - y^\intercal\right) \left\{\Phi_f^\dagger(x)\, |x-y| - \frac{1}{4} \int dv\, \Phi_f^\dagger(v)\, \epsilon(x-v)\, \epsilon(v-y)\right\} \end{aligned} \qquad (17)$$

Conclusions

We have careful performed the constraint analysis of the non-abelian $SQED_4$ where we have shown that: the the Gauss' law constraints is automatically conserved and the full set of constraints were determine. We observed that the non-abelian $SQED_4$ has a first class constraint, similar what happen with the abelian $SQED_4$ case [6], which result of a linear combination of constraints and it is the eigenvector of the constraints matrix with zero mode. It fact is a consequence of the existence of a set of constraints associated with the scalar sector. Finally, choosing the null-plane gauge we fix the first class constraints and by construction we obtained generalized Dirac brackets of the canonical variables. Our results are equivalents, after quantization, with these reported in the literature [6] [7] when the abelian case is considered.

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