

# Revised Criteria for Stability in the General Two-Higgs Doublet Model

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**Abstract.** We will revise one of the methods given in the literature to determine the necessary and sufficient conditions that the parameters must satisfy to have a stable scalar potential in the general two-Higgs doublet model. We will give a procedure that facilitates finding the conditions for stability of a scalar potential. The stability guarantees that the scalar potential has a global minimum, that is, the potential is bounded from below, which is a necessary condition to implement the spontaneous gauge-symmetry breaking in the models.

## 1. Revised criteria for stability

We obtain the stationary points of  $J_4(\mathbf{k})$  using Eq. (46) of Ref. [1] as follows:

$$(E - u)\mathbf{k} = -\boldsymbol{\eta}, \quad \text{with } |\mathbf{k}| = 1, \quad (1)$$

where  $|\mathbf{k}| < 1$  for the case  $u = 0$ . Now, suppose we find two solutions  $\mathbf{p}$  and  $\mathbf{q}$  with their respective Lagrange multipliers  $u_p$  and  $u_q$  such that

$$\begin{aligned} (E - u_p)\mathbf{p} &= -\boldsymbol{\eta}, \\ (E - u_q)\mathbf{q} &= -\boldsymbol{\eta}, \end{aligned} \quad (2)$$

where

$$|\mathbf{p}| = 1, \quad |\mathbf{q}| = 1 \quad \text{and} \quad u_p \neq u_q. \quad (3)$$

Let us evaluate the function  $J_4(\mathbf{k})$  at these stationary points

$$\begin{aligned} J_4(\mathbf{p}) &= u_p + \eta_{00} + \boldsymbol{\eta}^T \mathbf{p}, \\ J_4(\mathbf{q}) &= u_q + \eta_{00} + \boldsymbol{\eta}^T \mathbf{q}. \end{aligned} \quad (4)$$

Given that the matrix  $E$  is symmetric, from Eq. (2), we have

$$(u_q - u_p)\mathbf{p}^T \mathbf{q} = \boldsymbol{\eta}^T (\mathbf{p} - \mathbf{q}), \quad (5)$$

and taking into account (4), we obtain

$$J_4(\mathbf{p}) - J_4(\mathbf{q}) = (u_p - u_q)(1 - \mathbf{p}^T \mathbf{q}). \quad (6)$$

The product  $\mathbf{p}^T \mathbf{q} = |\mathbf{p}| |\mathbf{q}| \cos \theta = \cos \theta < 1$ . According to (3), the case  $\theta = 0$  implies that  $\mathbf{p} = \mathbf{q}$ , and from (5), we deduce  $u_p = u_q$ , which contradicts the assumed in (3). Further, the inequality  $\mathbf{p}^T \mathbf{q} < 1$  is immediately satisfied if  $|\mathbf{p}| < 1$ . Therefore, in any case, it is true that the factor  $(1 - \mathbf{p}^T \mathbf{q}) > 0$ , and consequently, from (6), we conclude that

$$u_p < u_q \iff J_4(\mathbf{p}) < J_4(\mathbf{q}). \quad (7)$$

The result (7) is quite useful because it makes it easier to find the conditions of the parameters to have a stable scalar potential. The process would be as follows: compute all the ‘‘regular’’ Lagrange multipliers  $\{u_i\}$  ( $i \leq 6$ ), by solving equation (52) of Ref. [1]. Include in this set the ‘‘exceptional’’ solutions  $\{\mu_j\}$  ( $j \leq 3$ ), by solving the equation  $\det(E - u) = 0$ , omitting the values  $\mu_j$  for which the corresponding  $\eta_j \neq 0$ , on the basis that E is diagonal (as you can see from (1)). Finally, consider  $u = 0$  for solutions within the sphere  $|\mathbf{k}| < 1$ , and with them, form the set  $S = \{u_i, \mu_j, 0\}$ , which has at most ten elements.

The result (7) suggests taking the smallest value of  $S$  to establish a stable scalar potential. Since the values of  $S$  are in general free parameters, let us assume that each one of them is the lowest value.

If the smallest value is a regular solution  $\{u_i\}$ , immediately impose the condition  $J_4(\mathbf{p}_i) > 0$ , that is,  $f(u_i) > 0$  (Eq. (51) in Ref. [1]), that which, according to the result (7), would guarantee the stability of the scalar potential. Conditions coming from regular solutions are necessary. Let us keep the regular solutions in  $S$ .

If the smallest value is an exceptional solution,  $\{\mu_j\}$ , you must first verify that it gives a valid stationary point, that is,  $f'(\mu_j) \geq 0$ . If this is not right, you can discard this value from the set  $S$ . In the case of being satisfied, impose the condition  $f(\mu_j) > 0$ , which would guarantee the stability of the potential according to the result (7). The conditions arising from the exceptional solutions may not be necessary since the inequality  $f'(\mu_j) \geq 0$  is not always satisfied. Similarly, if the smallest value of  $S$  is 0, you should check first that  $f'(0) > 0$ ; if not, discard this value from  $S$ . If it is satisfied, set the condition  $f(0) > 0$  to ensure the stability of the scalar potential.

Values of  $S$  that, given their structure, cannot be the smallest, are discarded if the lowest value gave a valid stationary point (according to the result (7)). Otherwise, they should be analyzed.

So far, the conditions above give stability in a ‘‘strong’’ sense. If for one of the cases above we have  $f(u) = 0$ , proceed as indicated in Ref. [1, 2], considering, in this case,  $J_2(\mathbf{k})$ , which would guarantee the stability of the scalar potential in the weak or marginal sense. For the remaining stationary points, it follows that  $J_4(\mathbf{k}) > 0$ , as stated in (7).

Finally, we build the set

$$I = \{\text{values not discarded from } S\}, \quad (8)$$

from which we obtain the sufficient conditions to guarantee the stability of the scalar potential. Let us apply the results above to a particular model.

## 2. Example: Stability for THDM

Let us analyze the two-Higgs-doublet model (THDM) of Gunion et al., with the Higgs potential given in Eq.(79) of Ref. [1]. After examining the potential, the corresponding Lagrange multipliers, including 0, which could result in possible stability conditions, give the following set:

$$S = \left\{ u_1 = \frac{1}{4}(2\lambda_1 - \lambda_4), u_2 = \frac{1}{4}(2\lambda_2 - \lambda_4), u_3 = 0, \mu_4 = \frac{1}{4}(\kappa - \lambda_4), \right. \\ \left. \mu_5 = \frac{1}{8} \left( -2\lambda_4 + \lambda_5 + \lambda_6 + \sqrt{(\lambda_5 - \lambda_6)^2 + \lambda_7^2} \right) \right\}, \quad (9)$$

where  $\kappa = \frac{1}{2} \left( \lambda_5 + \lambda_6 - \sqrt{(\lambda_5 - \lambda_6)^2 + \lambda_7^2} \right)$ . The first two parameters are the regular Lagrange multipliers, and the last two are the appropriate exceptional solutions in  $S$ . Note that  $\mu_4 < \mu_5$ , but we still cannot discard  $\mu_5$  since we must first check if  $f'(\mu_4) \geq 0$ . The global minimum of  $J_4(\mathbf{k})$  occurs where the minimum valid value of  $S$  is.

(i) If  $u_1$  is the smallest value of  $S$  in (9), then

$$f(u_1) > 0 \implies \lambda_1 + \lambda_3 > 0. \quad (10)$$

(ii) If  $u_2$  is the smallest value of  $S$  in (9), then

$$f(u_2) > 0 \implies \lambda_2 + \lambda_3 > 0. \quad (11)$$

Since  $u_1$  and  $u_2$  are regular solutions, the inequalities (10) and (11) are necessary.

(iii)

$$\text{If } u_3 = 0 < u_1, u_2, \mu_4, \mu_5, \quad (12)$$

we can observe that

$$f'(u_3) = \frac{4u_1u_2}{(u_1 + u_2)^2} > 0, \quad (13)$$

so  $u_3$  is not discarded. Taking into account the inequalities (10) and (11) in  $f(u_3)$ , we have

$$f(u_3) = \frac{\left[ -\lambda_4 - 2\lambda_3 + 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \right] \left[ \lambda_4 + 2\lambda_3 + 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \right]}{8(u_1 + u_2)} > 0, \quad (14)$$

and from Eq. (12) we can show that the factors  $u_1 + u_2 > 0$  and  $-\lambda_4 - 2\lambda_3 + 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} > 0$ ; therefore

$$\lambda_4 > -2\lambda_3 - 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}. \quad (15)$$

(iv)

$$\text{If } \mu_4 < u_1, u_2, u_3, \mu_5, \quad (16)$$

then

$$f'(\mu_4) = \frac{(2\lambda_1 - \kappa)(2\lambda_2 - \kappa)}{(\lambda_1 + \lambda_2 - \kappa)^2} > 0 \quad (17)$$

because of inequalities  $(2\lambda_1 - \kappa) > 0$ ,  $(2\lambda_2 - \kappa) > 0$  and  $(\lambda_1 + \lambda_2 - \kappa) > 0$  derived from the Eq. (16). So, the Lagrange multiplier  $\mu_4$  must be included in the set  $I$ . Besides,

$$f(\mu_4) = \frac{\left[ -\kappa - 2\lambda_3 + 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \right] \left[ \kappa + 2\lambda_3 + 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \right]}{4(\lambda_1 + \lambda_2 - \kappa)} > 0, \quad (18)$$

and using (16), we can show that the factors  $(\lambda_1 + \lambda_2 - \kappa) > 0$  and  $-\kappa - 2\lambda_3 + 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} > 0$ ; therefore

$$\kappa > -2\lambda_3 - 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}. \quad (19)$$

So the Lagrange multiplier  $\mu_5$  is not considered since  $\mu_5 > \mu_4$ .

In short, for the THDM to be stable, the following conditions on the parameters are *sufficient*

$$\lambda_1 + \lambda_3 > 0, \quad \lambda_2 + \lambda_3 > 0, \quad \lambda_4, \kappa > -2\lambda_3 - 2\sqrt{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}. \quad (20)$$

### 3. Conclusions

We can see that the application of the result (7) is essential to get a consistent model and be able to derive sufficient conditions to have a stable scalar potential. It allows us to identify either necessary conditions (for regular solutions) or conditions that may not be necessary, coming from exceptional solutions (including 0). Both conditions generate sufficient inequalities that guarantee the stability of a scalar potential. As an example, we can appreciate it, in the expression (152) of Ref. [1], where  $u_2 < u_1, u_3$ , so for stability conditions, only  $u_2$  is considered. In this sense, it may happen that some Lagrange multipliers, although not being the smallest values, must be taken into account for stability conditions. You can appreciate it from Gunion's potential in Sect. 2 (Eq. (79) of Ref. [1]), since if  $\mu_4$  were not a valid stationary point, we would have had to analyze  $\mu_5$ . In that way, we can reduce the number of sufficient conditions arising from exceptional solutions (including 0) provided that  $f'(\mu_j) < 0$  (or  $f'(0) \leq 0$ ).

### References

- [1] Maniatis M, von Manteuffel A, Nachtmann O and Nagel F 2006 *Eur. Phys. J.* **C48** 805–823 (*Preprint hep-ph/0605184*)
- [2] Nagel F 2004 *New aspects of gauge-boson couplings and the Higgs sector* Ph.D. thesis Heidelberg U. URL <http://www.ub.uni-heidelberg.de/archiv/4803>