Texture Zeros and WB Transformations in the Quark Sector of the Standard Model

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Stimulated by the recent attention given to the texture zeros found in the quark mass matrices sector of the Standard Model, an analytical method for identifying (or to exclude) texture zeros models will be implemented here, starting from arbitrary quark mass matrices and making a suitable weak basis (WB) transformation, we are be able to find equivalent quark mass matrix. It is shown that the number of non-equivalent quark mass matrix representations is finite. We give exact numerical results for parallel and non-parallel four-texture zeros models. We find that some five-texture zeros Ansätze are in agreement with all present experimental data. And we confirm definitely that six-texture zeros of Hermitian quark mass matrices are not viable models anymore.

I. INTRODUCTION

Although the gauge sector of the Standard Model (SM) with the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ symmetry is very successful, the Yukawa sector of the SM is still poorly understood. The origin of the fermion masses, the mixing angles and the CP violation remain as open problems in particle physics. There have been a lot of studies of possible fundamental symmetries in the Yukawa coupling matrices of the SM [1–3]. In the absence of a more fundamental theory of interactions, an independent phenomenological model approach to search for possible textures or symmetries in the fermion mass matrices is still playing an important role.

In the SM, the mass term is given by

$$-\mathcal{L}_M = \bar{u}_R M_u u_L + \bar{d}_R M_d d_L + h.c, \tag{1.1}$$

where the mass matrices M_u and M_d are three-dimensional complex matrices. In the most general case, they contain 36 real parameters. A first simplification, without losing generality, is by making use of the polar decomposition theorem of matrix algebra, by which, one can always express a general mass matrix as a product of a hermitian and unitary matrix. Therefore, we can consider quark mass matrices to be hermitian as the unitary matrix can be absorbed in the right handed quark fields. This immediately brings down the number of free parameters from 36 to 18.

A simple and instructive ansatz of hermitian quark mass matrices with six-texture zeros was first proposed in reference [1]. An additional non-parallel six-texture zeros was given in [4]. Both textures are currently ruled out [5], because, among other things, they do not reproduce some entries of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix V. Specifically, in both cases, the magnitude of $|V_{ub}/V_{cb}|$ predicted by $\sqrt{m_u/m_c}$ is too low ($V_{ub}/V_{cb} \approx 0.06$ or smaller for reasonable values of the quark masses m_u and m_c [6, 8]) to agree with the present experimental result ($|V_{ub}/V_{cb}|_{\rm ex} \approx 0.09$ [6]). Because of this, some authors have highly recommended the use of four-texture zeros [5, 9, 10]. It is shown in this work, that four texture zeros is readily feasible, and, we can even get five-zeros textures.

We would therefore present an analytical method to calculate models containing various texture zeros in the quark mass matrix sector, taking into account the latest experimental data provided [6]. We use simultaneously, in our research, two very common approach: one approach consists of placing zeros (called texture zeros) at certain entries of quark mass matrices that can predict self-consistent and experimentally-favored relations between quark masses and flavor mixing parameters [4, 11, 12]; which is used in conjunction with the other approach, the WB transformation (weak basis transformation), that transforms the quark mass matrix representations into new equivalent ones [9].

This paper is organized as follows: in Sect. II we discuss some issues related to the WB transformation method and its utilities. We dedicate, in Sect. III, to obtain some numerical parallel and non-parallel four-texture zeros quark mass matrices using special techniques for that; which we use, in Sect. IV, to find five-texture zeros in quark mass matrices compatible with the present experimental data; this configuration, is studied from an analytical point of view, in Sect. V; and our conclusions are presented in Sect. VI. And the method used extensively throughout this paper to find texture zeros is verified in Appendix A.

II. WB TRANSFORMATIONS

The most general WB transformation [9], that leaves the physical content invariable and the mass matrices Hermitian, is

$$\begin{aligned} M_u &\longrightarrow M_u' = U^\dagger M_u U, \\ M_d &\longrightarrow M_d' = U^\dagger M_d U, \end{aligned} \tag{2.1}$$

where U is an arbitrary unitary matrix. We say that the two representations $M_{u,d}$ and $M'_{u,d}$ are equivalent each other. Besides, it implies that the number of equivalent representations is infinity. This kind of transformation will be used extensively in calculations below.

But, firstly, let us show that the WB transformation is exhaustive in generating all possible mass matrix representations. Let us first consider the representation of Hermitian quark mass matrices indicated by (M_u, M_d), and diagonalize them as follows

$$U_u^\dagger M_u U_u = D_u \quad \text{and} \quad U_d^\dagger M_d U_d = D_d. \tag{2.2}$$

The CKM mixing matrix is given by

$$V_{ckm} = U_u^\dagger U_d. \tag{2.3}$$

On the other hand, the prime representation (M_u', M_d') gives

$$U_u'^{\dagger} M_u' U_u' = D_u \quad \text{and} \quad U_d'^{\dagger} M_d' U_d' = D_d, \tag{2.4}$$

and

$$V_{ckm} = U_u'^{\dagger} U_d'. \tag{2.5}$$

Equating the expressions (2.3) and (2.5) yields

$$U_u^{\dagger} U_d = U_u'^{\dagger} U_d' \Rightarrow U_u' U_u^{\dagger} = U_d' U_d^{\dagger}. \tag{2.6}$$

And equating (2.2) and (2.4), gives respectively

$$U_u'^{\dagger} M_u' U_u' = U_u^{\dagger} M_u U_u \quad \text{and} \quad U_d'^{\dagger} M_d' U_d' = U_d^{\dagger} M_d U_d, \tag{2.7}$$

where we find that the mass matrices M_u and M_d can be expressed in terms of the mass matrices M_u' and M_d' as follows

$$M_u = U_u U_u'^{\dagger} M_u' U_u' U_u^{\dagger}, \tag{2.8}$$
$$M_d = U_d U_d'^{\dagger} M_d' U_d' U_d^{\dagger}. \tag{2.9}$$

Using (2.6) into (2.9), we have

$$M_d = U_u U_u'^{\dagger} M_d' U_u' U_u^{\dagger}. \tag{2.10}$$

where $U = U_u U_u'^{\dagger}$ is an unitary matrix which allows us to state.

"In the SM, any two pairs of Hermitian quark mass matrices, given by (M_u, M_d) and (M_u', M_d'), with identical eigenvalues and flavor mixing parameters, to a specific scale energy, are related through a WB transformation," (2.11)

i.e., there is no a quark mass matrix representation outside the set (2.1). In this reasoning, we have assumed that both representations generates the same entries, including the phases, for the CKM mixing matrix (V_{ckm}); something valid due that a WB transformation makes them equal, as will be shown in section (II A).

The importance of the WB transformation, as calculation tool, can be appreciated from the following results.

A. The preliminary matrix representation

In the quark-family basis, it is more convenient to use the following quark mass matrix representation [9, 13]

$$\begin{aligned} M_u &= D_u = \begin{pmatrix} \lambda_{1u} & 0 & 0 \\ 0 & \lambda_{2u} & 0 \\ 0 & 0 & \lambda_{3u} \end{pmatrix}, \\ M_d &= V D_d V^{\dagger}, \end{aligned} \tag{2.12}$$

which comes from a WB transformation, and we call it as the *the u-diagonal representation*. We call the other possibility

$$\begin{aligned} M_u &= V^{\dagger} D_u V, \\ M_d &= D_d = \begin{pmatrix} \lambda_{1d} & 0 & 0 \\ 0 & \lambda_{2d} & 0 \\ 0 & 0 & \lambda_{3d} \end{pmatrix}, \end{aligned} \tag{2.13}$$

as *the d-diagonal representation*. One advantage of using representations (2.12) (or (2.13)) is to be able to use simultaneously the CKM mixing matrix V and the quark mass eigenvalues $|\lambda_{iu,d}|$ $(i = 1, 2, 3)$. Where $\lambda_{iu,d}$ may be either positive or negative and satisfy the hierarchy

$$|\lambda_{1u,d}| \ll |\lambda_{2u,d}| \ll |\lambda_{3u,d}|. \tag{2.14}$$

It is usually said that the CKM matrix is an arbitrary unitary matrix with five phases rotated away through the phase redefinition of the left handed up and down quark fields [14]. This can be shown by using the following unitary matrix

$$\begin{pmatrix} e^{ix} & & \\ & e^{iy} & \\ & & 1 \end{pmatrix}$$

in order to make a WB transformation on (2.12). The up matrix

$$M_u = \begin{pmatrix} e^{ix} & & \\ & e^{iy} & \\ & & 1 \end{pmatrix} D_u \begin{pmatrix} e^{ix} & & \\ & e^{iy} & \\ & & 1 \end{pmatrix}^\dagger = D_u, \tag{2.15}$$

remains equal, while the down matrix takes the form

$$M_d = \begin{pmatrix} e^{ix} & & \\ & e^{iy} & \\ & & 1 \end{pmatrix} \left(V D_d V^\dagger\right) \begin{pmatrix} e^{ix} & & \\ & e^{iy} & \\ & & 1 \end{pmatrix}^\dagger, \tag{2.16}$$

$$M_d = \left[\begin{pmatrix} e^{ix} & & \\ & e^{iy} & \\ & & 1 \end{pmatrix} V \begin{pmatrix} e^{i\alpha_1} & & \\ & e^{i\alpha_2} & \\ & & e^{i\alpha_3} \end{pmatrix}\right] D_d \left[\begin{pmatrix} e^{ix} & & \\ & e^{iy} & \\ & & 1 \end{pmatrix} V \begin{pmatrix} e^{i\alpha_1} & & \\ & e^{i\alpha_2} & \\ & & e^{i\alpha_3} \end{pmatrix}\right]^\dagger, \tag{2.17}$$

where in the last step we have used the identity (2.15) applied to the diagonal down mass matrix. The expression into the square brackets is precisely the most general way to write an unitary matrix [14].

In this representation, the matrix M_d, in (2.16), contains two free parameters x and y, which plays an important role to obtain texture zeros as we shall see later.

B. A unique negative eigenvalue

The result (2.11) permits us to use the u-diagonal representation (2.12) (or the d-diagonal representation (2.13)) as the starting point, to generate any other representation. If they exist, by this method, important texture zeros in mass matrix can be found.

Because some texture zeros must lie along its diagonal entries of both up and down Hermitian quark mass matrices, it implies that at least one and at most two of its eigenvalues must be negative [9]. Furthermore, for the case of two negative eigenvalues, these mass matrices can be reduced to have only one negative eigenvalue, by factoring a minus sign out which can be included, for instance, into the mass matrix basis (2.12). Thus, without loss of generality, the texture zeros models can be deduced considering that

"each one of quark mass matrices M_u and M_d contains exactly one negative eigenvalue." (2.18)

III. NUMERICAL FOUR-TEXTURE ZEROS

There are a wide variety of four-texture zeros representations. Using a specific approach, some non-parallel texture are easy to obtain. But more laborious methods are required in parallel cases. In our analysis we will use the next physical quantities.

A. Quark masses and CKM

For quark mass matrix phenomenology, values of $m_q(\mu)$ at $\mu = m_Z$ are useful, because the observed CKM matrix parameters $|V_{ij}|$ are given at $\mu = m_Z$. We summarize quark masses at $\mu = m_Z$ [7, 8, 13].

$$m_u = 1.38^{+0.42}_{-0.41}\,,\ m_c = 638^{+43}_{-84},\ m_t = 172100 \pm 1200\,,$$
$$m_d = 2.82 \pm 0.48\,,\ m_s = 57^{+18}_{-12}\,,\ m_b = 2860^{+160}_{-60}\,. \tag{3.1}$$

given in units of MeV.

The Cabibbo-Kobayashi-Maskawa (CKM) matrix [8, 15, 16] is a 3×3 unitary matrix,

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix},$$

which can be parametrized by three mixing angles and the CP-violating Kobayashi-Maskawa (KM) phase [16]. Of the many possible conventions, a standard choice has become [17]

$$V = \begin{pmatrix} c_{12}\,c_{13} & s_{12}\,c_{13} & s_{13}\,e^{-i\delta} \\ -s_{12}\,c_{23} - c_{12}\,s_{23}\,s_{13}\,e^{i\delta} & c_{12}\,c_{23} - s_{12}\,s_{23}\,s_{13}\,e^{i\delta} & s_{23}\,c_{13} \\ s_{12}\,s_{23} - c_{12}\,c_{23}\,s_{13}\,e^{i\delta} & -c_{12}\,s_{23} - s_{12}\,c_{23}\,s_{13}\,e^{i\delta} & c_{23}\,c_{13} \end{pmatrix}, \tag{3.2}$$

where $s_{ij} = \sin\theta_{ij}, c_{ij} = \cos\theta_{ij}$, and δ is the phase responsible for all CP-violating phenomena in flavor-changing processes in the SM. The angles θ_{ij} can be chosen to lie in the first quadrant, so $s_{ij}, c_{ij} \geq 0$.

It is known experimentally that $s_{13} \ll s_{23} \ll s_{12} \ll 1$, and it is convenient to exhibit this hierarchy using the Wolfenstein parametrization. We define [18, 19]

$$s_{12} = \lambda, \qquad s_{23} = A\,\lambda^2,$$
$$s_{13}\,e^{i\delta} = \frac{A\,\lambda^3(\bar\rho + i\,\bar\eta)\sqrt{1 - A^2\lambda^4}}{\sqrt{1-\lambda^2}\,[1 - A^2\,\lambda^4(\bar\rho + i\,\bar\eta)]}. \tag{3.3}$$

The constraints implied by the unitarity of the three generation CKM matrix significantly reduce the allowed range of some of the CKM elements. The fit for the Wolfenstein parameters defined in Eq. (3.3) gives

$$\lambda = 0.22535 \pm 0.00065, \quad A = 0.811^{+0.022}_{-0.012},$$
$$\bar\rho = 0.131^{+0.026}_{-0.013}, \quad \bar\eta = 0.345^{+0.013}_{-0.014}. \tag{3.4}$$

These values are obtained using the method of Refs. [18, 20]. The fit results for the values of all nine CKM elements are.

$$V = \begin{pmatrix} 0.974272 & 0.225349 & 0.00351322\,e^{-i\,1.20849} \\ 0.225209\,e^{-i\,3.14101} & 0.97344\,e^{-i\,3.13212\times10^{-5}} & 0.0411845 \\ 0.00867944\,e^{-i\,0.377339} & 0.0404125\,e^{-i\,3.12329} & 0.999145 \end{pmatrix}, \tag{3.5}$$

with magnitudes

$$|V| = \begin{pmatrix} 0.97427 \pm 0.00015 & 0.22534 \pm 0.00065 & 0.00351^{+0.00015}_{-0.00014} \\ 0.22520 \pm 0.00065 & 0.97344 \pm 0.00016 & 0.0412^{+0.0011}_{-0.0005} \\ 0.00867^{+0.00029}_{-0.00031} & 0.0404^{+0.0011}_{-0.0005} & 0.999146^{+0.000021}_{-0.000046} \end{pmatrix}, \tag{3.6}$$

and the Jarlskog invariant is

$$J = \left(2.96^{+0.20}_{-0.16}\right) \times 10^{-5}. \tag{3.7}$$

B. Non-parallel four-texture zeros

It is the most simple case. For instance, let us take the eigenvalues signs pattern as follow

$$\lambda_{1u} = -m_u, \lambda_{2u} = m_c, \lambda_{3u} = m_t, \tag{3.8}$$
$$\lambda_{1d} = m_d, \lambda_{2d} = -m_s, \lambda_{3d} = m_b. \tag{3.9}$$

Then, for this case, the numerical values in the u-diagonal representation (2.12) are

$$\begin{aligned} M_u &= \begin{pmatrix} -1.38 & & \\ & 638 & \\ & & 172100 \end{pmatrix} \text{MeV}, \\ M_d &= \begin{pmatrix} -0.2 \pm 0.8 & -12.9758 - 0.386978i & 4.09941 - 9.38819i \\ -12.9758 + 0.386978i & -49.0183 & 119.924 - 0.043146i \\ 4.09941 + 9.38819i & 119.924 + 0.043146i & 2855.02 \end{pmatrix} \text{MeV}, \\ &= \begin{pmatrix} 0 & -12.9758 - 0.386978i & 4.09941 - 9.38819i \\ -12.9758 + 0.386978i & -49.0183 & 119.924 - 0.043146i \\ 4.09941 + 9.38819i & 119.924 + 0.043146i & 2855.02 \end{pmatrix} \text{MeV}, \end{aligned} \tag{3.10}$$

where we have used the numerical CKM matrix (3.5) and errors of (3.6). In the second mass matrix above, in the entry $M_d(1,1) = -0.2 \pm 0.8$ calculated, since the uncertainty (± 0.8) in determining this element exceeds the value of 0.2 it is obviously reasonable to call the $(1,1)$ entry zero ($M_d(1,1) = 0$). Something pointed out in Reference [13].

Making a WB transformation on (3.10) using the following unitary matrix

$$U = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{3.11}$$

with $\tan\theta = \sqrt{\frac{m_u}{m_t}}$, the matrices (3.10) transform into a form, where the entries $(1,1)$, $(1,2)$ and $(2,3)$ of matrix M_u becomes zero. Then, we have

$$M'_u = U M_u U^\dagger = \begin{pmatrix} 0 & 0 & 487.338 \\ 0 & 638 & 0 \\ 487.338 & 0 & 172099 \end{pmatrix} \text{MeV}, \tag{3.12}$$

and

$$\begin{aligned} M'_d &= U M_d U^\dagger \\ &= \begin{pmatrix} 0 & -12.6361 - 0.386854i & 12.1844 - 9.38819i \\ -12.6361 + 0.386854i & -49.0183 & 119.96 - 0.0442417i \\ 12.1844 + 9.38819i & 119.96 + 0.0442417i & 2854.97 \end{pmatrix} \text{MeV}, \end{aligned} \tag{3.13}$$

where the element $M'_d(1,1)$ is zero for the same reason given in (3.10). We finally obtain a non-parallel four-texture zeros mass matrix representation.

$$\begin{aligned} M'_u &= \begin{pmatrix} 0 & 0 & 487.338 \\ 0 & 638 & 0 \\ 487.338 & 0 & 172099 \end{pmatrix} \text{MeV}, \\ M'_d &= \begin{pmatrix} 0 & 12.6421\, e^{-3.11099i} & 15.3817\, e^{-0.656498i} \\ 12.6421 e^{3.11099i} & -49.0183 & 119.96\, e^{-0.000368804i} \\ 15.3817\, e^{0.656498i} & 119.96\, e^{0.000368804i} & 2854.97 \end{pmatrix} \text{MeV}. \end{aligned} \tag{3.14}$$

New equivalent four-texture zeros representations can be obtained using the former representation. For example, if we use unitary matrices looking like

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.15}$$

and apply them to (3.14), it allows us to obtain new non-parallel four-texture zeros representations. For the case (3.15), we have

$$\begin{aligned} M_u &= \begin{pmatrix} 0 & 487.338 & 0 \\ 487.338 & 172099 & 0 \\ 0 & 0 & 638 \end{pmatrix} \text{MeV}, \\ M_d &= \begin{pmatrix} 0 & 15.3817\, e^{-0.656498i} & 12.6421\, e^{-3.11099i} \\ 15.3817\, e^{0.656498i} & 2854.97 & 119.96\, e^{0.000368804i} \\ 12.6421 e^{3.11099i} & 119.96\, e^{-0.000368804i} & -49.0183 \end{pmatrix} \text{MeV}. \end{aligned} \tag{3.16}$$

where some of their entries have been permuted.

We have found typical non-parallel four-texture zeros quark mass matrix representations. The WB was applied by using simple unitary matrices like (3.11). The process is more difficult if we want to find parallel texture zeros in quark mass matrices.

C. Parallel four-texture zeros

Let us begin implementing a method that we shall apply later to special cases. Let us start by giving the following structure for the up matrix elements [1]

$$M_u = \begin{pmatrix} 0 & |C_u| & 0 \\ |C_u| & \tilde{B}_u & |B_u| \\ 0 & |B_u| & A_u \end{pmatrix}, \tag{3.17}$$

where $\tilde{B}_u$ and A_u are real numbers. The mass matrix M_u can be diagonalized using the transformation

$$O_u^\dagger M_u O_u = \begin{pmatrix} \lambda_{1u} & & \\ & \lambda_{2u} & \\ & & \lambda_{3u} \end{pmatrix}, \tag{3.18}$$

where the exact analytical result of O_u is [5]

$$O_u = \begin{pmatrix} e^{ix}\rho\sqrt{\frac{\lambda_{2u}\lambda_{3u}(A_u-\lambda_{1u})}{A_u(\lambda_{2u}-\lambda_{1u})(\lambda_{3u}-\lambda_{1u})}} & e^{iy}\eta\sqrt{\frac{\lambda_{1u}\lambda_{3u}(\lambda_{2u}-A_u)}{A_u(\lambda_{2u}-\lambda_{1u})(\lambda_{3u}-\lambda_{2u})}} & \sqrt{\frac{\lambda_{1u}\lambda_{2u}(A_u-\lambda_{3u})}{A_u(\lambda_{3u}-\lambda_{1u})(\lambda_{3u}-\lambda_{2u})}} \\ -e^{ix}\eta\sqrt{\frac{\lambda_{1u}(\lambda_{1u}-A_u)}{(\lambda_{2u}-\lambda_{1u})(\lambda_{3u}-\lambda_{1u})}} & e^{iy}\sqrt{\frac{\lambda_{2u}(A_u-\lambda_{2u})}{(\lambda_{2u}-\lambda_{1u})(\lambda_{3u}-\lambda_{2u})}} & \rho\sqrt{\frac{\lambda_{3u}(\lambda_{3u}-A_u)}{(\lambda_{3u}-\lambda_{1u})(\lambda_{3u}-\lambda_{2u})}} \\ e^{ix}\eta\sqrt{\frac{\lambda_{1u}(A_u-\lambda_{2u})(A_u-\lambda_{3u})}{A_u(\lambda_{2u}-\lambda_{1u})(\lambda_{3u}-\lambda_{1u})}} & -e^{iy}\rho\sqrt{\frac{\lambda_{2u}(A_u-\lambda_{1u})(\lambda_{3u}-A_u)}{A_u(\lambda_{2u}-\lambda_{1u})(\lambda_{3u}-\lambda_{2u})}} & \sqrt{\frac{\lambda_{3u}(A_u-\lambda_{1u})(A_u-\lambda_{2u})}{A_u(\lambda_{3u}-\lambda_{1u})(\lambda_{3u}-\lambda_{2u})}} \end{pmatrix}, \tag{3.19}$$

where $\eta \equiv \lambda_{2u}/m_c = +1$ or -1 and $\rho \equiv \lambda_{3u}/m_t = +1$ or -1 corresponding to the possibility $(\lambda_{1u}, \lambda_{2u}, \lambda_{3u}) = (-m_u, m_c, m_t)$, $(\lambda_{1u}, \lambda_{2u}, \lambda_{3u}) = (m_u, -m_c, m_t)$ or $(\lambda_{1u}, \lambda_{2u}, \lambda_{3u}) = (m_u, m_c, -m_t)$. The arbitrary phase factors in (3.19) were included, in order that given them appropriated values, the generated CKM matrix becomes compatible with the chosen convention (3.2) [2]. Note that $\tilde{B}_u$, $|B_u|$ and $|C_u|$ can be expressed in terms of λ_{iu} $(i = 1, 2, 3)$ and A_u, using invariant matrix functions as follows

$$\mathrm{tr} M_u \Rightarrow \tilde{B}_u = \lambda_{1u} + \lambda_{2u} + \lambda_{3u} - A_u, \tag{3.20}$$

$$\mathrm{tr} M_u^2 \Rightarrow |B_u| = \sqrt{\frac{(A_u - \lambda_{1u})(A_u - \lambda_{2u})(\lambda_{3u} - A_u)}{A_u}}, \tag{3.21}$$

$$\det M_u \Rightarrow |C_u| = \sqrt{\frac{-\lambda_{1u}\lambda_{2u}\lambda_{3u}}{A_u}}, \tag{3.22}$$

where "tr" and "det" are the trace and the determinant respectively. The matrix O_u can be seen as the unitary matrix such that the WB transformation transforms the representation (2.12) into the form

$$M_u' = O_u \begin{pmatrix} \lambda_{1u} & & \\ & \lambda_{2u} & \\ & & \lambda_{3u} \end{pmatrix} O_u^\dagger = \begin{pmatrix} 0 & |C_u| & 0 \\ |C_u| & \tilde{B}_u & |B_u| \\ 0 & |B_u| & A_u \end{pmatrix}, \tag{3.23}$$

$$M_d' = O_u (V D_d V^\dagger) O_u^\dagger = \begin{pmatrix} X_{(A_u,x,y)} & C_d & Y_{(A_u,x,y)} \\ C_d^* & \tilde{B}_d & B_d \\ Y_{(A_u,x,y)}^* & B_d^* & A_d \end{pmatrix}, \tag{3.24}$$

where the elements of M_d' depends on three parameters A_u, x and y. To complete the analysis, we must obtain neglected values at the entries $(1,1)$ and $(1,3)$ compared with the remaining elements of the matrix M_d'. Then we have to solve three equations

$$X_{(A_u,x,y)} = 0, \ \mathrm{Re}[Y_{(A_u,x,y)}] = 0, \text{ and } \mathrm{Im}[Y_{(A_u,x,y)}] = 0 \tag{3.25}$$

where "Re" refers to the real part and "Im" the imaginary part of the function. In the process the following details must be taken into account:

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[1] It is sufficient to consider that the mass matrix be real and symmetric, since the phases may be included later by means of a WB process.
[2] It is not necessary to include a phase factor in the third column of O_u, since we can factor out it.

- The formulas (3.20) through (3.22) must be real numbers. Therefore, the parameter A_u is restricted to lie into an interval. Let us see the different possibilities
  - If $\lambda_{1u} = -m_u$, $\lambda_{2u} = m_c$ and $\lambda_{3u} = m_t$ then
$$m_c < A_u < m_t. \tag{3.26}$$
  - If $\lambda_{1u} = m_u$, $\lambda_{2u} = -m_c$ and $\lambda_{3u} = m_t$ then
$$m_u < A_u < m_t. \tag{3.27}$$
  - If $\lambda_{1u} = m_u$, $\lambda_{2u} = m_c$ and $\lambda_{3u} = -m_t$ then
$$m_u < A_u < m_c. \tag{3.28}$$

  where the hierarchy (2.14) was considered.
- The phases given in (3.19) could have been included initially in the transformation (2.16), instead to write them explicitly in the matrix O_u. The validity of this point of view is checked by observing that the matrix (3.19) can be decomposed as the product of two matrices, where the right hand side contains the phases as follows
$$O_u = O_{u(x=0,y=0)} \begin{pmatrix} e^{ix} & & \\ & e^{iy} & \\ & & 1 \end{pmatrix}, \tag{3.29}$$
  such that, after replacing this decomposition into (3.24) and comparing with (2.16), we conclude that both points of view concur.

In appendix A, we will work a case previously studied in the paper [9] and replicate the results presented there by using the techniques implemented here.

1. *Example 1: parallel four-texture zeros*

We are mainly concerned to find four-texture zeros with the recent data given in Section III A. Let us take the following case
$$\lambda_{1u} = -m_u, \lambda_{2u} = m_c, \lambda_{3u} = m_t, \tag{3.30}$$
$$\lambda_{1d} = -m_d, \lambda_{2d} = m_s, \lambda_{3d} = m_b. \tag{3.31}$$

We have, in the u-diagonal representation, the following mass matrix representation.
$$\begin{aligned} M_u &= \begin{pmatrix} -1.38 & 0 & 0 \\ 0 & 638 & 0 \\ 0 & 0 & 172100 \end{pmatrix} \text{MeV}, \\ M_d &= V D_d V^\dagger \\ &= \begin{pmatrix} 0.253114 & 13.2691 - 0.386919i & 3.01706 - 9.38676i \\ 13.2691 + 0.386919i & 58.7203 & 115.45 + 0.043146i \\ 3.01706 + 9.38676i & 115.45 - 0.043146i & 2855.21 \end{pmatrix} \text{MeV}. \end{aligned} \tag{3.32}$$

Making a WB transformation on (3.32), using the unitary matrix O_u (Eq. 3.19), the following conditions
$$\begin{aligned} M'_{d(1,1)}(A_u, x_1, x_2, y_1, y_2) &= 0, \\ \text{Re}\left[M'_{d(1,3)}(A_u, x_1, x_2, y_1, y_2)\right] &= 0, \\ \text{Im}\left[M'_{d(1,3)}(A_u, x_1, x_2, y_1, y_2)\right] &= 0, \end{aligned} \tag{3.33}$$
are established, in order to find zero entries in (1,1), (1,3) and (3,1) of the resulting matrix $M'_d = O_u M_d O_u^\dagger$; where the phases given in O_u has been defined as $e^{ix} = \cos x + i \sin x = x_1 + ix_2$ and $e^{iy} = \cos y + i \sin y = y_1 + iy_2$, such that
$$x_1^2 + x_2^2 = 1 \quad \text{and} \quad y_1^2 + y_2^2 = 1. \tag{3.34}$$

Eqs. (3.33) and (3.34) gives the following exact solution.

$$\begin{aligned} A_u &= 153231 \text{ MeV},\ x_1 = 0.883194,\ x_2 = -0.469007, \\ y_1 &= 0.202996, \quad y_2 = 0.97918. \end{aligned} \tag{3.35}$$

Finally, we obtain an exact parallel four-texture zeros mass matrix representation.

$$M'_u = O_u M_u O_u^\dagger = \begin{pmatrix} 0 & 31.4461 & 0 \\ 31.4461 & 19505.7 & 53659.2 \\ 0 & 53659.2 & 153231 \end{pmatrix} \text{MeV}, \tag{3.36}$$

$$\begin{aligned} M'_d &= O_u M_d O_u^\dagger, \\ &= \begin{pmatrix} 0 & -1.43578 - 13.3956i & 0 \\ -1.43578 + 13.3956i & 381.367 & 893.365 + 113.383i \\ 0 & 893.365 - 113.383i & 2532.81 \end{pmatrix} \text{MeV}, \end{aligned} \tag{3.37}$$

In the same way, we can find other non-equivalent parallel four-texture zeros representations. Let us look another case.

2. *Example 2: another parallel four-texture zeros model*

Another possibility that works well is

$$\lambda_{1u} = m_u, \lambda_{2u} = m_c, \lambda_{3u} = -m_t, \tag{3.38}$$

$$\lambda_{1d} = m_d, \lambda_{2d} = m_s, \lambda_{3d} = -m_b, \tag{3.39}$$

from which, we have $A_u = 7.34102$ MeV$, x_1 = 0.998393$, $x_2 = -0.0566637, y_1 = 0.999664$ and $y_2 = 0.0259074$. Thus, the corresponding parallel four-texture zeros mass matrix representation is

$$M'_u = O_u M_u O_u^\dagger = \begin{pmatrix} 0 & 4543.2 & 0 \\ 4543.2 & -171468. & 9388.13 \\ 0 & 9388.13 & 7.34102 \end{pmatrix} \text{MeV}, \tag{3.40}$$

$$\begin{aligned} M'_d &= O_u M_d O_u^\dagger \\ &= \begin{pmatrix} 0 & 123.93 + 10.0184i & 0 \\ 123.93 - 10.0184i & -2829.92 & 267.035 + 1.39152i \\ 0 & 267.035 - 1.39152i & 29.738 \end{pmatrix} \text{MeV}. \end{aligned} \tag{3.41}$$

IV. NUMERICAL FIVE-TEXTURE ZEROS

Now, let us try to find five-texture zeros for the quark mass matrix sector. If this cannot be achieved, we can conclude that five and six-texture zeros are not viable models. For that, we will use the mathematical tools previously implemented in Sect. III C. We shall begin as usual by proposing a texture zeros configuration, in this case with three zeros for the up/down quark mass matrix[3], and see how many zeros can be reached for the down/up quark mass matrix. In principle, there are many possibilities, but many of them are equivalent ones. In total, there are two non-equivalent cases, depending on the number of zeros included in their diagonal entries. Therefore, we have only two possibilities: one-zero or two-zero in diagonal entries. Let us name them as *one-zero family* and *two-zero family*, respectively. With an appropriated unitary matrix and performing the corresponding WB transformation the other possibilities are obtained. In the Table I both families are indicated, which summarizes the equivalent possibilities for each case. Let us study each family.

A. Two-zero family

In what follows, we work the *u-diagonal* and *d-diagonal* cases simultaneously. The standard representation for the two-zero family is

$$M_{u,d} = \begin{pmatrix} 0 & |C_{u,d}| & 0 \\ |C_{u,d}| & 0 & |B_{u,d}| \\ 0 & |B_{u,d}| & A_{u,d} \end{pmatrix}. \tag{4.1}$$

[3] A model with four zeros in the up/down quark mass matrix is not realistic.

Unitary matrix	**Two-zero Family** $(p_i\, M_{u,d}\, p_i^T)$	**One-zero family** $(p_i\, M_{u,d}\, p_i^T)$
$p_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \lvert C_{u,d}\rvert & 0 \\ \lvert C_{u,d}\rvert & 0 & \lvert B_{u,d}\rvert \\ 0 & \lvert B_{u,d}\rvert & A_{u,d} \end{pmatrix}$	$\begin{pmatrix} 0 & \lvert B_{u,d}\rvert & 0 \\ \lvert B_{u,d}\rvert & C_{u,d} & 0 \\ 0 & 0 & A_{u,d} \end{pmatrix}$
$p_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \lvert C_{u,d}\rvert \\ 0 & A_{u,d} & \lvert B_{u,d}\rvert \\ \lvert C_{u,d}\rvert & \lvert B_{u,d}\rvert & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \lvert B_{u,d}\rvert \\ 0 & A_{u,d} & 0 \\ \lvert B_{u,d}\rvert & 0 & C_{u,d} \end{pmatrix}$
$p_3 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$	$\begin{pmatrix} A_{u,d} & \lvert B_{u,d}\rvert & 0 \\ \lvert B_{u,d}\rvert & 0 & \lvert C_{u,d}\rvert \\ 0 & \lvert C_{u,d}\rvert & 0 \end{pmatrix}$	$\begin{pmatrix} A_{u,d} & 0 & 0 \\ 0 & C_{u,d} & \lvert B_{u,d}\rvert \\ 0 & \lvert B_{u,d}\rvert & 0 \end{pmatrix}$
$p_4 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \lvert C_{u,d}\rvert & \lvert B_{u,d}\rvert \\ \lvert C_{u,d}\rvert & 0 & 0 \\ \lvert B_{u,d}\rvert & 0 & A_{u,d} \end{pmatrix}$	$\begin{pmatrix} \lvert C_{u,d}\rvert & \lvert B_{u,d}\rvert & 0 \\ \lvert B_{u,d}\rvert & 0 & 0 \\ 0 & 0 & A_{u,d} \end{pmatrix}$
$p_5 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$	$\begin{pmatrix} A_{u,d} & 0 & \lvert B_{u,d}\rvert \\ 0 & 0 & \lvert C_{u,d}\rvert \\ \lvert B_{u,d}\rvert & \lvert C_{u,d}\rvert & 0 \end{pmatrix}$	$\begin{pmatrix} A_{u,d} & 0 & 0 \\ 0 & 0 & \lvert B_{u,d}\rvert \\ 0 & \lvert B_{u,d}\rvert & C_{u,d} \end{pmatrix}$
$p_6 = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$	$\begin{pmatrix} 0 & \lvert B_{u,d}\rvert & \lvert C_{u,d}\rvert \\ \lvert B_{u,d}\rvert & A_{u,d} & 0 \\ \lvert C_{u,d}\rvert & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} C_{u,d} & 0 & \lvert B_{u,d}\rvert \\ 0 & A_{u,d} & 0 \\ \lvert B_{u,d}\rvert & 0 & 0 \end{pmatrix}$

TABLE I. One and two-zero Family.

and its diagonalization matrix satisfies the following relation

$$O_{u,d}^{\dagger} M_{u,d} O_{u,d} = \begin{pmatrix} \lambda_{1u,d} & & \\ & \lambda_{2u,d} & \\ & & \lambda_{3u,d} \end{pmatrix}, \tag{4.2}$$

where one and only one $\lambda_{iu,d}$ is assumed to be a negative number. The invariant quantities "det" and "trace" applied on (4.1) and (4.2)

$$\mathrm{tr} M_{u,d} = A_{u,d} = \lambda_{1u,d} + \lambda_{2u,d} + \lambda_{3u,d}, \tag{4.3}$$

$$\begin{aligned} \mathrm{tr} M_{u,d}^2 &= A_{u,d}^2 + 2|B_{u,d}|^2 + 2|C_{u,d}|^2 \\ &= \lambda_{1u,d}^2 + \lambda_{2u,d}^2 + \lambda_{3u,d}^2, \end{aligned} \tag{4.4}$$

$$\det M_{u,d} = -A_{u,d}|C_{u,d}|^2 = \lambda_{1u,d}\lambda_{2u,d}\lambda_{3u,d}, \tag{4.5}$$

allow us to express the parameters of (4.1) in terms of its eigenvalues

$$A_{u,d} = \lambda_{1u,d} + \lambda_{2u,d} + \lambda_{3u,d}, \tag{4.6}$$

$$|B_{u,d}| = \sqrt{-\frac{(\lambda_{1u,d} + \lambda_{2u,d})(\lambda_{1u,d} + \lambda_{3u,d})(\lambda_{2u,d} + \lambda_{3u,d})}{A_{u,d}}}, \tag{4.7}$$

$$|C_{u,d}| = \sqrt{-\frac{\lambda_{1u,d}\lambda_{2u,d}\lambda_{3u,d}}{A_{u,d}}}. \tag{4.8}$$

From expression (4.8), together with (2.18), we have that

$$A_{u,d} > 0, \tag{4.9}$$

and using (4.7) and the hierarchy (2.14) we found that only one possibility is permitted

$$\lambda_{1u,d}, \lambda_{3u,d} > 0 \quad \text{and} \quad \lambda_{2u,d} < 0. \tag{4.10}$$

For the u-diagonal case, the diagonalization matrix (3.19) becomes

$$O_u = \begin{pmatrix} 0.99892e^{ix} & -0.0464583e^{iy} & 0.0000104863 \\ 0.0463719e^{ix} & 0.997078e^{iy} & 0.0607083 \\ -0.00283086e^{ix} & -0.0606422e^{iy} & 0.998156 \end{pmatrix}, \tag{4.11}$$

and for the d-diagonal case, the diagonalization matrix is given by

$$O_d = \begin{pmatrix} 0.980856e^{ix} & -0.194731e^{iy} & 0.000682127 \\ 0.19251e^{ix} & 0.970182e^{iy} & 0.147267 \\ -0.0293392e^{ix} & -0.144316e^{iy} & 0.989097 \end{pmatrix}. \tag{4.12}$$

As you can see, in both cases, we are treating with quasi diagonal matrices.

Performing the WB transformation using the unitary matrix $O_{u,d}$ we have

$$M'_{u,d} = O_{u,d} \begin{pmatrix} \lambda_{1u,d} & & \\ & \lambda_{2u,d} & \\ & & \lambda_{3u,d} \end{pmatrix} O^{\dagger}_{u,d}\,, \tag{4.13}$$

$$= \begin{pmatrix} 0 & |C_{u,d}| & 0 \\ |C_{u,d}| & 0 & |B_{u,d}| \\ 0 & |B_{u,d}| & A_{u,d} \end{pmatrix} \quad \text{and} \tag{4.14}$$

$$M'_{d,u} = O_{d,u} M_{d,u} O^{\dagger}_{d,u}, \tag{4.15}$$

where the matrices

$$M_d = V D_d V^{\dagger} \quad \text{and} \quad M_u = V^{\dagger} D_u V, \tag{4.16}$$

depend on if we work with either *the u-diagonal* or *the d-diagonal case.*

In order to facilitate the calculus we define the following new variables

$$\begin{aligned} e^{ix} &= x_1 + ix_2, \text{ with } \quad x_1^2 + x_2^2 = 1, \\ e^{iy} &= y_1 + iy_2, \text{ with } \quad y_1^2 + y_2^2 = 1, \end{aligned} \tag{4.17}$$

where their norms satisfy

$$|x_1|, |x_2| \leq 1, \quad \text{and} \quad |y_1|, |y_2| \leq 1. \tag{4.18}$$

With the former definitions, the elements of the matrix $M'_{d,u}$ defined in (4.15) have now a polynomial form in each case considered: $\lambda_{1d} = -m_d$ or $\lambda_{2d} = -m_s$ or $\lambda_{3d} = -m_b$ for the u-digonal case (or $\lambda_{1u} = -m_d$ or $\lambda_{2u} = -m_s$ or $\lambda_{3u} = -m_b$ for the d-diagonal case) . The results are summarized in Tables (II) and (III)

1. *Analysis of "down" mass matrix.*

Table (II) summarizes the components of M'_d for *the u-diagonal case.* By simple inspection, using (4.18), shows that is not possible to find zeros at entries (2,2), (2,3) and (3,3). And not solutions were found for either

$$\begin{aligned} &\mathrm{Re}[M'_d(1,2)] = 0, \quad \mathrm{Im}[M'_d(1,2)] = 0, \quad \text{or} \\ &\mathrm{Re}[M'_d(1,3)] = 0, \quad \mathrm{Im}[M'_d(1,3)] = 0, \end{aligned}$$

equations. Therefore, it is impossible to find two texture zeros into the down quark mass matrix coming from an u-diagonal representation for the two-zero family case.

2. *Analysis of "up" mass matrix and a model with five-texture zeros.*

Let us consider *the d-diagonal case.* The entries of matrix M'_u, after the WB transformation is made, are given in the Table (III). According to the Table, only entries (1,2) and (1,3) deserve some attention. From which, only the cases $\lambda_{1u} = -m_u$ and $\lambda_{2u} = -m_c$ give an acceptable solution.

M'_d	Negative mass eigenvalue		
entries	Case 1. $\lambda_{1d} = -m_d$ (MeV)	Case 2. $\lambda_{2d} = -m_s$ (MeV)	Case 3. $\lambda_{3d} = -m_b$ (MeV)
$M'_d(1,1)$	$0.758616 + 0.0000632072x_1 + 0.000196652x_2 - 0.000112489y_1 - 1.23159x_1y_1 - 0.0359124x_2y_1 + 0.0359124x_1y_2 - 1.23159x_2y_2$	$-0.575839 + 0.0000858823x_1 + 0.000196682x_2 - 0.000116848y_1 + 1.20436x_1y_1 - 0.0359178x_2y_1 + 0.0359178x_1y_2 + 1.20436x_2y_2$	$11.261 - 0.0000849534x_1 - 0.00019705x_2 + 0.000116858y_1 - 1.0895x_1y_1 + 0.0359851x_2y_1 - 0.0359851x_1y_2 - 1.0895x_2y_2$
$M'_d(1,2)$	$-5.41488 + 0.182964x_1 + 0.569243x_2 - 0.324408y_1 + 13.1875x_1y_1 + 0.384538x_2y_1 + 0.000121238y_2 - 0.384538x_1y_2 + 13.1875x_2y_2 + i(-0.569234x_1 + 0.182961x_2 - 0.00012214y_1 - 0.386206x_1y_1 + 13.2447x_2y_1 - 0.326823y_2 - 13.2447x_1y_2 - 0.386206x_2y_2)$	$4.52621 + 0.248601x_1 + 0.56933x_2 - 0.336979y_1 - 12.8959x_1y_1 + 0.384597x_2y_1 - 0.000121238y_2 - 0.384597x_1y_2 - 12.8959x_2y_2 + i(-0.569321x_1 + 0.248597x_2 + 0.00012214y_1 - 0.386264x_1y_1 - 12.9519x_2y_1 - 0.339487y_2 + 12.9519x_1y_2 - 0.386264x_2y_2)$	$-4.05674 - 0.245913x_1 - 0.570396x_2 + 0.337008y_1 + 11.6661x_1y_1 - 0.385317x_2y_1 + 0.000109807y_2 + 0.385317x_1y_2 + 11.6661x_2y_2 + i(0.570386x_1 - 0.245909x_2 - 0.000110625y_1 + 0.386987x_1y_1 + 11.7166x_2y_1 + 0.339516y_2 - 11.7166x_1y_2 + 0.386987x_2y_2)$
$M'_d(1,3)$	$0.359323 + 3.00824x_1 + 9.35933x_2 - 5.35379y_1 - 0.802056x_1y_1 - 0.0233874x_2y_1 + 0.00200082y_2 + 0.0233874x_1y_2 - 0.802056x_2y_2 + i(-9.35933x_1 + 3.00824x_2 - 0.00200077y_1 + 0.0234892x_1y_1 - 0.805546x_2y_1 - 5.35364y_2 + 0.805546x_1y_2 + 0.0234892x_2y_2)$	$-0.245286 + 4.08743x_1 + 9.36075x_2 - 5.56125y_1 + 0.784325x_1y_1 - 0.023391x_2y_1 - 0.00200082y_2 + 0.023391x_1y_2 + 0.784325x_2y_2 + i(-9.36075x_1 + 4.08743x_2 + 0.00200077y_1 + 0.0234928x_1y_1 + 0.787738x_2y_1 - 5.56109y_2 - 0.787738x_1y_2 + 0.0234928x_2y_2)$	$0.216621 - 4.04322x_1 - 9.37828x_2 + 5.56172y_1 - 0.709524x_1y_1 + 0.0234348x_2y_1 + 0.00181218y_2 - 0.0234348x_1y_2 - 0.709524x_2y_2 + i(9.37828x_1 - 4.04322x_2 - 0.00181213y_1 - 0.0235368x_1y_1 - 0.712611x_2y_1 + 5.56157y_2 + 0.712611x_1y_2 - 0.0235368x_2y_2)$
$M'_d(2,2)$	$127.279 + 0.016987x_1 + 0.0528505x_2 + 13.9766y_1 + 1.22703x_1y_1 + 0.0357795x_2y_1 - 0.00522333y_2 - 0.0357795x_1y_2 + 1.22703x_2y_2$	$-86.9431 + 0.023081x_1 + 0.0528585x_2 + 14.5182y_1 - 1.19991x_1y_1 + 0.035785x_2y_1 + 0.00522333y_2 - 0.035785x_1y_2 - 1.19991x_2y_2$	$87.5349 - 0.0228314x_1 - 0.0529575x_2 - 14.5194y_1 + 1.08547x_1y_1 - 0.035852x_2y_1 - 0.00473086y_2 + 0.035852x_1y_2 + 1.08547x_2y_2$
$M'_d(2,3)$	$165.914 + 0.13913x_1 + 0.432866x_2 + 114.475y_1 - 0.0747673x_1y_1 - 0.00218017x_2y_1 - 0.0427818y_2 + 0.00218017x_1y_2 - 0.0747673x_2y_2 + i(-0.436093x_1 + 0.140167x_2 + 0.0430994y_1 + 0.000139144x_2y_1 + 115.325y_2 - 0.000139144x_1y_2)$	$178.932 + 0.189042x_1 + 0.432932x_2 + 118.911y_1 + 0.0731144x_1y_1 - 0.0021805x_2y_1 + 0.0427818y_2 + 0.0021805x_1y_2 + 0.0731144x_2y_2 + i(-0.436159x_1 + 0.190451x_2 - 0.0430994y_1 - 0.000136068x_2y_1 + 119.794y_2 + 0.000136068x_1y_2)$	$-178.968 - 0.186998x_1 - 0.433742x_2 - 118.921y_1 - 0.0661415x_1y_1 + 0.00218458x_2y_1 - 0.0387482y_2 - 0.00218458x_1y_2 - 0.0661415x_2y_2 + i(0.436975x_1 - 0.188392x_2 + 0.0390359y_1 + 0.000123091x_2y_1 - 119.804y_2 - 0.000123091x_1y_2)$
$M'_d(3,3)$	$2845.12 - 0.0170502x_1 - 0.0530471x_2 - 13.9765y_1 + 0.00455581x_1y_1 + 0.000132844x_2y_1 + 0.00522329y_2 - 0.000132844x_1y_2 + 0.00455581x_2y_2$	$2844.14 - 0.0231669x_1 - 0.0530552x_2 - 14.5181y_1 - 0.00445509x_1y_1 + 0.000132865x_2y_1 - 0.00522329y_2 - 0.000132865x_1y_2 - 0.00445509x_2y_2$	$-2844.14 + 0.0229163x_1 + 0.0531545x_2 + 14.5193y_1 + 0.00403021x_1y_1 - 0.000133113x_2y_1 + 0.00473083y_2 + 0.000133113x_1y_2 + 0.00403021x_2y_2$

TABLE II. *The u-diagonal representation*: the "down" mass matrix entries for the two-zero family case.

For the first case, with $\lambda_{1u} = -m_u$, we have

$$M'_u(1,2) = 0, \tag{4.19}$$

$$M'_u(1,1) \approx 0, \tag{4.20}$$

where

$$\begin{aligned} x_1 &= 0.706984, \;\; y_1 = -0.540778, \\ x_2 &= 0.70723, \;\; y_2 = -0.841165. \end{aligned} \tag{4.21}$$

The corresponding five-texture zeros representation obtained, is.

$$M'_u = \begin{pmatrix} 0 & 0 & -92.3618 + 157.694i \\ 0 & 5748.17 & 28555.1 + 5911.83i \\ -92.3618 - 157.694i & 28555.1 - 5911.83i & 166988 \end{pmatrix} \text{MeV}, \tag{4.22a}$$

$$M'_d = \begin{pmatrix} 0 & 13.9899 & 0 \\ 13.9899 & 0 & 424.808 \\ 0 & 424.808 & 2796.9 \end{pmatrix} \text{MeV}. \tag{4.22b}$$

M'_u entries	Negative mass eigenvalue Case 1. $\lambda_{1u} = -m_u$ (MeV)	Case 2. $\lambda_{2u} = -m_c$ (MeV)	Case 3. $\lambda_{3u} = -m_t$ (MeV)
$M'_u(1,1)$	$151.93 + 1.84869x_1 - 0.735842x_2 + 1.839y_1 + 74.8244x_1y_1 - 8.85442x_2y_1 + 0.0337901y_2 + 8.85442x_1y_2 + 74.8244x_2y_2$	$-59.245 + 1.86453x_1 - 0.735821x_2 + 1.85259y_1 - 32.2673x_1y_1 - 8.92032x_2y_1 + 0.0337892y_2 + 8.92032x_1y_2 - 32.2673x_2y_2$	$64.2966 - 1.86453x_1 + 0.735833x_2 - 1.85259y_1 + 32.0358x_1y_1 + 8.92032x_2y_1 - 0.0337897y_2 - 8.92032x_1y_2 + 32.0358x_2y_2$
$M'_u(1,2)$	$-300.727 + 199.742x_1 - 79.504x_2 + 193.933y_1 - 179.051x_1y_1 + 21.1882x_2y_1 + 3.56336y_2 - 21.1882x_1y_2 - 179.051x_2y_2 + i(79.3596x_1 + 199.379x_2 - 3.73171y_1 - 22.926x_1y_1 - 193.736x_2y_1 + 203.095y_2 + 193.736x_1y_2 - 22.926x_2y_2)$	$132.633 + 201.453x_1 - 79.5017x_2 + 195.366y_1 + 77.2138x_1y_1 + 21.3458x_2y_1 + 3.56326y_2 - 21.3458x_1y_2 + 77.2138x_2y_2 + i(79.3573x_1 + 201.087x_2 - 3.7316y_1 - 23.0966x_1y_1 + 83.5468x_2y_1 + 204.596y_2 - 83.5468x_1y_2 - 23.0966x_2y_2)$	$-131.697 - 201.453x_1 + 79.503x_2 - 195.366y_1 - 76.6599x_1y_1 - 21.3458x_2y_1 - 3.56332y_2 + 21.3458x_1y_2 - 76.6599x_2y_2 + i(-79.3586x_1 - 201.087x_2 + 3.73166y_1 + 23.0966x_1y_1 - 82.9474x_2y_1 - 204.596y_2 + 82.9474x_1y_2 + 23.0966x_2y_2)$
$M'_u(1,3)$	$163.157 + 1340.29x_1 - 533.481x_2 + 1333.97y_1 + 26.6073x_1y_1 - 3.1486x_2y_1 + 24.5107y_2 + 3.1486x_1y_2 + 26.6073x_2y_2 + i(533.503x_1 + 1340.34x_2 - 24.4856y_1 + 3.41346x_1y_1 + 28.8455x_2y_1 + 1332.61y_2 - 28.8455x_1y_2 + 3.41346x_2y_2)$	$98.7777 + 1351.77x_1 - 533.466x_2 + 1343.83y_1 - 11.4741x_1y_1 - 3.17203x_2y_1 + 24.51y_2 + 3.17203x_1y_2 - 11.4741x_2y_2 + i(533.488x_1 + 1351.83x_2 - 24.4849y_1 + 3.43886x_1y_1 - 12.4393x_2y_1 + 1342.46y_2 + 12.4393x_1y_2 + 3.43886x_2y_2)$	$-98.9206 - 1351.77x_1 + 533.475x_2 - 1343.83y_1 + 11.3918x_1y_1 + 3.17203x_2y_1 - 24.5103y_2 - 3.17203x_1y_2 + 11.3918x_2y_2 + i(-533.497x_1 - 1351.83x_2 + 24.4853y_1 - 3.43886x_1y_1 + 12.3501x_2y_1 - 1342.46y_2 - 12.3501x_1y_2 - 3.43886x_2y_2)$
$M'_u(2,2)$	$5396.4 + 78.3341x_1 - 31.1797x_2 - 1978.06y_1 - 73.1657x_1y_1 + 8.65814x_2y_1 - 36.3452y_2 - 8.65814x_1y_2 - 73.1657x_2y_2$	$3115.84 + 79.0053x_1 - 31.1788x_2 - 1992.67y_1 + 31.552x_1y_1 + 8.72257x_2y_1 - 36.3441y_2 - 8.72257x_1y_2 + 31.552x_2y_2$	$-3115.38 - 79.0051x_1 + 31.1793x_2 + 1992.67y_1 - 31.3256x_1y_1 - 8.72257x_2y_1 + 36.3447y_2 + 8.72257x_1y_2 - 31.3256x_2y_2$
$M'_u(2,3)$	$24777.1 + 257.091x_1 - 102.331x_2 - 6495.55y_1 + 11.0171x_1y_1 - 1.30373x_2y_1 - 119.35y_2 + 1.30373x_1y_2 + 11.0171x_2y_2 + i(107.083x_1 + 269.029x_2 + 124.757y_1 - 0.0158108x_1y_1 - 0.13361x_2y_1 - 6789.79y_2 + 0.13361x_1y_2 - 0.0158108x_2y_2)$	$25116.1 + 259.294x_1 - 102.328x_2 - 6543.55y_1 - 4.75103x_1y_1 - 1.31343x_2y_1 - 119.347y_2 + 1.31343x_1y_2 - 4.75103x_2y_2 + i(107.08x_1 + 271.334x_2 + 124.753y_1 - 0.0159285x_1y_1 + 0.0576178x_2y_1 - 6839.96y_2 - 0.0576178x_1y_2 - 0.0159285x_2y_2)$	$-25116.1 - 259.293x_1 + 102.33x_2 + 6543.55y_1 + 4.71695x_1y_1 + 1.31343x_2y_1 + 119.349y_2 - 1.31343x_1y_2 + 4.71695x_2y_2 + i(-107.081x_1 - 271.334x_2 - 124.755y_1 + 0.0159285x_1y_1 - 0.0572044x_2y_1 + 6839.96y_2 + 0.0572044x_1y_2 + 0.0159285x_2y_2)$
$M'_u(3,3)$	$168118 - 80.1828x_1 + 31.9155x_2 + 1976.22y_1 - 1.6587x_1y_1 + 0.196283x_2y_1 + 36.3114y_2 - 0.196283x_1y_2 - 1.6587x_2y_2$	$168065. - 80.8698x_1 + 31.9146x_2 + 1990.82y_1 + 0.715295x_1y_1 + 0.197744x_2y_1 + 36.3103y_2 - 0.197744x_1y_2 + 0.715295x_2y_2$	$-168065. + 80.8696x_1 - 31.9151x_2 - 1990.82y_1 - 0.710164x_1y_1 - 0.197744x_2y_1 - 36.3109y_2 + 0.197744x_1y_2 - 0.710164x_2y_2$

TABLE III. *The d-diagonal representation:* the "up" mass matrix entries for the two-zero family case.

Other possibility that works well is the following numerical five-texture zeros in the two-zero family case.

$$M'_u = \begin{pmatrix} 0 & 0 & 123.038 - 285.496i \\ 0 & 1430.03 & 18632.8 - 2336.25i \\ 123.038 + 285.496i & 18632.8 + 2336.25i & 170033 \end{pmatrix} \text{MeV}, \tag{4.23a}$$

$$M'_d = \begin{pmatrix} 0 & 13.2473 & 0 \\ 13.2473 & 0 & 425.817 \\ 0 & 425.817 & 2796.6 \end{pmatrix} \text{MeV}, \tag{4.23b}$$

with $\lambda_{2u} = -m_c$.

B. One-zero family

A typical representation of this family is given by

$$M_{u,d} = \begin{pmatrix} 0 & |B_{u,d}| & 0 \\ |B_{u,d}| & C_{u,d} & 0 \\ 0 & 0 & A_{u,d} \end{pmatrix}. \tag{4.24}$$

The mass matrix $M_{u,d}$ is diagonalized as follows

$$O^{\dagger}_{u,d} M_{u,d} O_{u,d} = O^{\dagger}_{u,d} \begin{pmatrix} 0 & |B_{u,d}| & 0 \\ |B_{u,d}| & C_{u,d} & 0 \\ 0 & 0 & A_{u,d} \end{pmatrix} O_{u,d} \,, \tag{4.25}$$

$$= \begin{pmatrix} \lambda_{1u,d} & & \\ & \lambda_{2u,d} & \\ & & \lambda_{3u,d} \end{pmatrix} , \tag{4.26}$$

The following matricial functions allows us to write the elements of $M_{u,d}$ in terms of its eigenvalues $\lambda_{iu,d}$. They are

$$\mathrm{tr} M_{u,d} = A_{u,d} + C_{u,d} = \lambda_{1u,d} + \lambda_{2u,d} + \lambda_{3u,d} \,, \tag{4.27}$$

$$\begin{aligned} \mathrm{tr} M^2_{u,d} &= A^2_{u,d} + 2|B_{u,d}|^2 + C^2_{u,d} \,, \\ &= \lambda^2_{1u,d} + \lambda^2_{2u,d} + \lambda^2_{3u,d} \,, \end{aligned} \tag{4.28}$$

$$\det M_{u,d} = -A_{u,d}|B_{u,d}|^2 = \lambda_{1u,d}\lambda_{2u,d}\lambda_{3u,d} \,, \tag{4.29}$$

from which we have various solutions

a):

$$\begin{aligned} &A_{u,d} = \lambda_{1u,d}, \quad |B_{u,d}| = \sqrt{-\lambda_{2u,d}\lambda_{3u,d}} \,, \\ &C_{u,d} = \lambda_{2u,d} + \lambda_{3u,d} \,, \end{aligned} \tag{4.30}$$

b):

$$\begin{aligned} &A_{u,d} = \lambda_{2u,d}, \quad |B_{u,d}| = \sqrt{-\lambda_{1u,d}\lambda_{3u,d}} \,, \\ &C_{u,d} = \lambda_{1u,d} + \lambda_{3u,d} \,, \end{aligned} \tag{4.31}$$

c):

$$\begin{aligned} &A_{u,d} = \lambda_{3u,d}, \quad |B_{u,d}| = \sqrt{-\lambda_{1u,d}\lambda_{2u,d}} \,, \\ &C_{u,d} = \lambda_{1u,d} + \lambda_{2u,d} \,. \end{aligned} \tag{4.32}$$

Each one of these former cases were analyzed. Both representations *u-diagonal* and *d-diagonal* were worked. The Eqs. (4.30, 4.31, 4.32), gives two possibilities for each case a), b) and c), depending of what eigenvalue is negative. In turn, each one of these cases, contain three possibilities depending of the negative eigenvalue assigned for the down (up) mass matrix . In total there are 36 possibilities. Neither of this cases were able to give models with five-texture or six-texture zeros.

V. ANALYTICAL FIVE-TEXTURE ZEROS AND THE CKM MATRIX

The five-texture zeros form of Eq. (4.23), derived under the conditions given in section IV A 2, is specially interesting because with the latest low energy data shows that it is a viable model, something not considered or rule out in papers like [5, 9, 21]. Therefore, let us assume the following five-texture zeros model

$$M_u = P^{\dagger} \begin{pmatrix} 0 & 0 & |C_u| \\ 0 & A_u & |B_u| \\ |C_u| & |B_u| & \tilde{B}_u \end{pmatrix} P, \quad M_d = \begin{pmatrix} 0 & |C_d| & 0 \\ |C_d| & 0 & |B_d| \\ 0 & |B_d| & A_d \end{pmatrix} , \tag{5.1}$$

where up and down quark mass matrices are given in the most general way, $P = \mathrm{diag}(e^{-i\phi_{c_u}}, e^{-i\phi_{b_u}}, 1)$ with $\phi_{b_u} \equiv \arg(B_u)$ and $\phi_{c_u} \equiv \arg(C_u)$, where the phases for M_d were no considered because they can be absorbed, through a WB transformation, into P. Considering $\lambda_{2u} = -m_c$, we have from (3.20) through (3.22) that

$$\begin{aligned} \tilde{B}_u &= m_u + m_t - m_c - A_u, \\ |B_u| &= \frac{\sqrt{A_u + m_c}\,\sqrt{m_t - A_u}\,\sqrt{A_u - m_u}}{\sqrt{A_u}}, \\ |C_u| &= \frac{\sqrt{m_c}\,\sqrt{m_t}\,\sqrt{m_u}}{\sqrt{A_u}}, \end{aligned} \tag{5.2}$$

where (3.27) was considered.

Taking into account (4.9) and (4.10), for the down mass matrix we have that

$$\begin{aligned} A_d &= m_d + m_b - m_s, \\ |B_d| &= \frac{\sqrt{m_d+m_b}\,\sqrt{m_b-m_s}\,\sqrt{m_s-m_d}}{\sqrt{m_d+m_b-m_s}}, \\ |C_d| &= \frac{\sqrt{m_b}\,\sqrt{m_d}\,\sqrt{m_s}}{\sqrt{m_d+m_b-m_s}}. \end{aligned} \tag{5.3}$$

The unitary matrix U_u which diagonalizes M_u is given by

$$U_u = P^\dagger \cdot p_2 \cdot O_u \approx \begin{pmatrix} \frac{\sqrt{Au-m_u}e^{i\,(\phi_{c_u}+x_u)}}{\sqrt{Au}} & -\frac{\sqrt{A_u+m_c}\,\sqrt{m_u}e^{i\,(\phi_{c_u}+y_u)}}{\sqrt{A_u}\,\sqrt{m_c}} & \frac{\sqrt{m_c}\,\sqrt{m_t-A_u}\,\sqrt{m_u}e^{i\,(\phi_{c_u}+z_u)}}{\sqrt{A_u}\,m_t} \\ -\frac{\sqrt{A_u+m_c}\,\sqrt{m_t-A_u}\,\sqrt{m_u}\,e^{i(\phi_{b_u}+x_u)}}{\sqrt{A_u}\,\sqrt{m_c}\,\sqrt{m_t}} & -\frac{\sqrt{m_t-A_u}\sqrt{A_u-m_u}\,e^{i(\phi_{b_u}+y_u)}}{\sqrt{m_t}\sqrt{A_u}} & \frac{\sqrt{A_u+m_c}\sqrt{A_u-m_u}\,e^{i(\phi_{b_u}+z_u)}}{\sqrt{m_t}\sqrt{A_u}} \\ \frac{\sqrt{A_u-m_u}\,\sqrt{m_u}\,e^{ix_u}}{\sqrt{m_c}\,\sqrt{m_t}} & \frac{\sqrt{A_u+m_c}\,e^{iy_u}}{\sqrt{m_t}} & \frac{\sqrt{m_t-A_u}\,e^{iz_u}}{\sqrt{m_t}} \end{pmatrix}, \tag{5.4}$$

where an additional phase factor e^{iz_u} in third column of O_u (Eq. (3.19)) were added, in order to reproduce all phases present in the CKM matrix. The 3×3 matrix $p_2 = [(1,0,0),(0,0,1),(0,1,0)]$ and the hierarchy (2.14) together with (3.27) were considered.

And the unitary matrix U_d which diagonalizes M_d is given by

$$U_d \approx \begin{pmatrix} e^{ix_d} & -\frac{\sqrt{m_d}\,e^{iy_d}}{\sqrt{m_s}} & \frac{\sqrt{m_d}\,m_s}{(m_b)^{3/2}} \\ \frac{\sqrt{m_d}\,e^{ix_d}}{\sqrt{m_s}} & e^{iy_d} & \frac{\sqrt{m_s}}{\sqrt{m_b}} \\ -\frac{\sqrt{m_d}\,e^{ix_d}}{\sqrt{m_b}} & -\frac{\sqrt{m_s}\,e^{iy_d}}{\sqrt{m_b}} & 1 \end{pmatrix}, \tag{5.5}$$

where, in the process, a phase factor in the third column was not necessary to be included. Now, we can easily find the CKM matrix $V = U_u^\dagger U_d$. In particular, using the matrix form (5.4) and (5.5) for U_u, U_d respectively, can survive current experimental tests. To leading order, we obtain.

$$|V_{ud}| \approx |V_{cs}| \approx |V_{tb}| \approx 1, \tag{5.6a}$$

$$|V_{us}| \approx |V_{cd}| \approx \left| \sqrt{\frac{A_u+m_c}{A_u}}\sqrt{\frac{m_u}{m_c}} + e^{\pm i(\phi_{b_u}-\phi_{c_u})}\sqrt{\frac{m_d}{m_s}} \right|, \tag{5.6b}$$

$$|V_{cb}| \approx |V_{ts}| \approx \left| \sqrt{\frac{m_s}{m_b}} - e^{\pm i\phi_{b_u}}\sqrt{\frac{A_u+m_c}{m_t}} \right|, \tag{5.6c}$$

$$\frac{|V_{ub}|}{|V_{cb}|} \approx \sqrt{\frac{m_u}{m_c}} \left| \frac{\sqrt{\frac{A_u}{m_t}} - e^{-i\phi_{b_u}}\sqrt{\frac{A_u+m_c}{A_u}}\sqrt{\frac{m_s}{m_b}}}{\sqrt{\frac{A_u+m_c}{m_t}} - e^{-i\phi_{b_u}}\sqrt{\frac{m_s}{m_b}}} \right|, \tag{5.6d}$$

$$\frac{|V_{td}|}{|V_{ts}|} \approx \sqrt{\frac{m_d}{m_s}}, \tag{5.6e}$$

where we assume that $m_u \ll A_u \ll m_t$. The sign "+" for V_{us}, V_{cb} and "−" for V_{cd}, V_{ts}. Note that if $A_u \gg m_c$ then $\frac{|V_{ub}|}{|V_{cb}|} \approx \sqrt{\frac{m_u}{m_c}}$, but this is not our case.

It is obvious that Eq. (5.6a), (5.6b) and (5.6e) are consistent with the previous results [5, 22]. A good fit of Eqs. (5.6) and the CKM to the experimental data suggests

$$\begin{gathered} A_u = 1430.03\,\text{MeV},\ \phi_{b_u} = -0.124733,\ \phi_{c_u} = -1.16389, \\ x_u = -1.83392,\ y_u = -2.68335,\ z_u = 0.00200664,\ x_d = -3.00697,\ y_d = 0.344676, \end{gathered} \tag{5.7}$$

which differ from the values given in [5, 22], $\phi_1 \approx \pi/3 \sim (\phi_{b_u} - \phi_{c_u})$, such that it is an important contribution term of CP-violation in the context of present mass matrices, and $\phi_2 \approx \pi/25 \sim -\phi_{b_u} \to 0$. The numerical analysis shows that, by plugging for the quark masses the values given in (3.1) and the input parameters in (5.7), we obtain the following absolute values for the mixing matrix

$$|V_{ckm}| = \begin{pmatrix} 0.993 & 0.255 \pm 0.030 & 0.00334 \pm 0.00094 \\ 0.255 \pm 0.030 & 1.004 & 0.034 \pm 0.014 \\ 0.0079 \pm 0.0020 & 0.035 \pm 0.014 & 1.011 \end{pmatrix}, \tag{5.8}$$

in good agreement with the experimental measured values presented in (3.6). For the Wolfenstein parameters we find that

$$\begin{aligned} \lambda' &= 0.247 \pm 0.027, \quad A' = 0.55^{+0.43}_{-0.31}, \\ \bar{\rho}' &= 0.117 \pm 0.061, \quad \bar{\eta}' = 0.361 \pm 0.070, \end{aligned} \tag{5.9}$$

which is in quite good agreement with the fit experimental values (3.4). The inner angles of the CKM unitarity triangle, $V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0$, are

$$\begin{aligned} \beta &= \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right) = 24.4114^\circ, \\ \alpha &= \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right) = 82.6294^\circ, \\ \gamma &= \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) = 72.9592^\circ, \end{aligned} \tag{5.10}$$

tha are into the constraint stablished by [8]. And the Jarlskog invariant obtained is

$$J' = \mathrm{Im}(V_{us}V_{ub}^*V_{cs}^*V_{cb}) = 2.8322 \times 10^{-5}, \tag{5.11}$$

which can be found into the interval given in (3.7).

VI. CONCLUSIONS

Within the Standard Model framework, we have investigated texture zeros for quark mass matrices that reproduce the quark masses and experimental mixing parameters. To simplify the problem, without loss of generality, we consider that the quark mass matrices are Hermitian, since the right chirality fields are singlets under the gauge symmetry SU(2). So, for any model where the fields are right chiral singlet under the local gauge symmetry, we may consider that their mass matrices are Hermitian. Specific six-texture zeros in quark mass matrices, including the Fritzsch model [1] and others like [4], have already been discarded because they can not adjust their results to the experimental data known at present. In Sect. II, together with the definition of WB transformation, it is shown that the number of non-equivalent representations for the quark mass matrices is finite, which greatly simplifies the problem. Through WB transformations was relatively easy to find non-parallel four-texture zeros mass matrices. More difficult, but feasible, was the case for parallel four-texture zeros mass matrices, which were found in an exact way. Significant was the consistent five-texture zeros quark mass matrix found by us. Similarly, we show the impossibility, under any circumstances, to find mass matrices with six-texture zeros consistent with experimental data. This is a generalization of six-texture zeros mass matrices discarded by Fritzsch et al.

Throughout this letter, into the SM, we have used the fact that all WB are equivalent. The opposite case is valid too, i.e., two quark mass matrices representations giving the same physical quantities must be related through a WB transformation. Which is condensed in statement (2.11).

By making appropriated WB transformations, numerical parallel and non parallel four-texture zeros were found. An exhaustive deduction process allows us to find a five-texture zeros numerical structure compatible with the experimental data, Eqs. (4.22) and (4.23). This representation was found in the two-zero family case. Equivalent representations are given in Table I.

We have determined the impossibility to find quark mass matrices having a total of six-texture zeros which are consistent with the measured values of the quark masses and mixing angles. While, a consistent model with five-texture zeros were successful. The five texture zero Ansatz of Eq.(5.1) (with $\lambda_{2u} = -m_c$), together with some assumptions which include appropriated values for A_u, ϕ_{b_u}, ϕ_{c_u}, x_u, y_u, z_u, x_d and y_d does lead to successful predictions for V_{CKM} such as those of Eqs.(5.6), (5.8), (5.9), (5.10) and (5.11) [4]. One nice thing about five-texture zeros quark mass matrices (5.1) is that no hierarchies on quark masses is necessary to be imposed to make correct predictions, although, expressions (5.6) become in a more complex notation.

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[4] In the case $\lambda_{1u} = -m_u$ similar results can be found.

Appendix A: Verification of the Method.

The paper [9] uses the following quark mass data:

$$m_u = 2.50 \text{ MeV}, m_c = 600 \text{ MeV}, m_t = 174000 \text{ MeV}, \tag{A1}$$
$$m_d = 4.00 \text{ MeV}, m_s = 80 \text{ MeV}, m_b = 3000 \text{ MeV}. \tag{A2}$$

and the numerical CKM matrix used is

$$V = \begin{pmatrix} 0.036195 + 0.97493i & -0.057798 + 0.21177i & 0.00037188 - 0.0035669i \\ -0.21247 + 0.054471i & 0.97351 + 0.050582i & -0.0044010 - 0.039760i \\ 0.0043605 + 0.0083871i & 0.0086356 - 0.038067i & 0.99836 + 0.040693i \end{pmatrix}. \tag{A3}$$

We assume the following case:

$$\lambda_{1u} = -m_u, \lambda_{2u} = m_c, \lambda_{3u} = m_t, \tag{A4}$$
$$\lambda_{1d} = -m_d, \lambda_{2d} = m_s, \lambda_{3d} = m_b. \tag{A5}$$

Then, the quark mass matrices (2.12) are

$$M_u = \begin{pmatrix} -2.5 & & \\ & 600 & \\ & & 174000 \end{pmatrix} \text{MeV}, \tag{A6}$$
$$M_d = \begin{pmatrix} 0.086447 & -3.4055 + 17.655i & -0.039835 - 10.774i \\ -3.4055 - 17.655i & 80.631 & -17.515 - 115.56i \\ -0.039835 + 10.774i & -17.515 + 115.56i & 2995.3 \end{pmatrix} \text{MeV}. \tag{A7}$$

Let us use the diagonalization matrix (3.19) with $x = \pi$ and $y = \pi$.

$$O_u \approx 10^{-3} \begin{pmatrix} -997.92 & -64.527\sqrt{\frac{A_u - 600}{A_u}} & 0.22297\sqrt{\frac{174000 - A_u}{A_u}} \\ 0.15442\sqrt{A_u} & -2.3965\sqrt{A_u - 600} & 2.4014\sqrt{174000 - A_u} \\ -0.15442\sqrt{\frac{(174000 - A_u)(A_u - 600)}{A_u}} & 2.3965\sqrt{174000 - A_u} & 2.4014\sqrt{A_u - 600} \end{pmatrix} \tag{A8}$$

where the approximation $A_u >> m_u$ was assumed because of the restriction (3.26). The matrix O_u now plays the role of a unitary matrix to make the WB transformation on (A6) and (A7). The entries, in the new representation, depend of A_u. In order to have a texture zeros at the entry $(1, 3)$, we need to solve

$$\begin{aligned} M_d(1,3) = Y(A_u) \propto\ & (8144.2 - 42221i)A_u\sqrt{174000 - A_u} + (95.463 + 25819i)A_u\sqrt{A_u - 600} - \\ & (10852)\sqrt{A_u(A_u - 600)(174000 - A_u)} - (9.3588 - 61.745i)(174000 - A_u)\sqrt{A_u} + \\ & (2714.0 + 17906i)(A_u - 600)\sqrt{A_u} + (0.0013716 - 0.37097i)(174000 - A_u)\sqrt{A_u - 600} \\ & - (33.934 + 175.92i)(A_u - 600)\sqrt{174000 - A_u} \approx 0, \end{aligned} \tag{A9}$$

whose solution is $A_u \approx 84621$ MeV, which agrees perfectly with the value given in the aforementioned paper. The quark mass matrices (A6) and (A7) take the form

$$M'_u = O_u M_u O_u^T = \begin{pmatrix} 0 & 55.537 & 0 \\ 55.537 & 89977 & 86660 \\ 0 & 86660 & 84621 \end{pmatrix} \text{MeV}, \tag{A10}$$
$$M'_d = O_u M_d O_u^T \tag{A11}$$
$$\approx \begin{pmatrix} 0 & 2.5792 + 25.325i & 0 \\ 2.5792 - 25.325i & 1600.5 & 1456.0 + 114.63i \\ 0 & 1456.0 - 114.63i & 1475.5 \end{pmatrix} \text{MeV}. \tag{A12}$$

At the present stage we have not yet obtained the matrices given in (25) and (26) of paper [9]. But we can make an additional WB transformation using the following diagonal unitary matrix with phase entries

$$P = \begin{pmatrix} 1 & & \\ & e^{i4.4984} & \\ & & e^{-i0.063300} \end{pmatrix}. \tag{A13}$$

We finally get the desired matrices

$$M''_u = P^\dagger M'_u P$$
$$= \begin{pmatrix} 0 & -11.794 - 54.270i & 0 \\ -11.794 + 54.270i & 89977 & -13009 + 85678i \\ 0 & -13009 - 85678i & 84621 \end{pmatrix} \text{MeV}, \tag{A14}$$

$$M''_d = P^\dagger M'_d P$$
$$= \begin{pmatrix} 0 & 24.199 - 7.8983i & 0 \\ 24.199 + 7.8983i & 1600.5 & -331.91 + 1422.3i \\ 0 & -331.91 - 1422.3i & 1475.5 \end{pmatrix} \text{MeV}. \tag{A15}$$

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