Yang-Mills Field on the Null-Plane

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Free Yang-Mills field

For any semi-simple Lie group with structure constants f^a_{bc} the Yang-Mill lagrangian density is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu}, \tag{1}$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{bc} A^b_\mu A^c_\nu$, the gauge index a, b, c runs from 1 to n.

In the present work, for convenience, we specialize in the $SU(2)$ gauge group. From (1) we find the Euler-Lagrange equations

$$(D_\nu)^{ab} F^{\nu\mu}_b = 0, \tag{2}$$

where we have defined the covariant derivative $(D_\nu)^{ab} \equiv \delta^a_b \partial_\nu - g\varepsilon_{abc} A^c_\nu$.

Structure Constraints and Classification

The null plane time x^+ and longitudinal coordinate x^- are defined, respectively, as

$$x^+ \equiv \frac{x^0 + x^3}{\sqrt{2}} \quad , \quad x^- \equiv \frac{x^0 - x^3}{\sqrt{2}}, \tag{3}$$

with the transverse coordinates $\underline{x} \equiv (x^1, x^2)$ kept unchanged. Hence, in the space of four-vectors $x^\mu = (x^+, x^1, x^2, x^-)$, the metric is

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{4}$$

Explicitly,

$$\begin{aligned} x^+ &= x_- \qquad , \qquad x^- = x_+, \\ x \cdot y &= x^+ y^- + x^- y^+ - x^k x^k, \end{aligned} \tag{5}$$

with $k = 1, 2$. The derivatives with respect to x^+ e x^- are defined as

$$\partial_+ \equiv \frac{\partial}{\partial x^+} \qquad , \qquad \partial_- \equiv \frac{\partial}{\partial x^-}. \tag{6}$$

with $\partial^+ = \partial_-$.

In the null-plane dynamics, the canonical conjugate momenta are

$$\pi_a^\mu \equiv \frac{\partial \mathcal{L}}{\partial \left(\partial_+ A_\mu^a\right)} = -F_a^{+\mu} \ , \tag{7}$$

this equation gives the following set of primary constraints

$$\phi_a \equiv \pi_a^+ \approx 0 \qquad , \qquad \phi_a^k \equiv \pi_a^k - \partial_- A_k^a + \partial_k A_-^a - g\varepsilon_{abc} A_-^b A_k^c \approx 0 \ . \tag{8}$$

and the dynamical relation for A_-^a

$$\pi_a^- = \partial_+ A_-^a - \partial_- A_+^a - g\varepsilon_{abc} A_+^b A_-^c \ , \tag{9}$$

We define the primary hamiltonian adding to the canonical hamiltonian the primary constraints

$$H_P = \int d^3y \left\{ \frac{1}{2} \left(\pi_a^-\right)^2 + \pi_a^- \left(D_-^x\right)^{ab} A_+^b + \pi_a^i \left(D_i^x\right)^{ab} A_+^b + \frac{1}{4} \left(F_{ij}^a\right)^2 + u^b \phi_b + \lambda_l^b \phi_b^l \right\}, \tag{10}$$

where u^b and λ_l^b are their respective Lagrange multipliers.

The fundamental Poisson brackets (PB) among fields are

$$\left\{ A_\mu^a(x), \pi_b^\nu(y) \right\} = \delta_\mu^\nu \delta_b^a \delta^3(x-y). \tag{11}$$

The consistency condition of the primary constraints give us

$$\begin{aligned} \dot{\phi}_a &= \{\phi_a, H_P\} = \left(D_{-}^{x}\right)^{ab} \pi_b^{-} + \left(D_i^x\right)^{ab} \pi_b^i \\ &\equiv G_a \approx 0 \;, \\ \dot{\phi}_a^k &= \left\{\phi_a^k, H_P\right\} = \left(D_k^x\right)^{ab} F_{+-}^b + \left(D_i^x\right)^{ab} F_{ik}^b - 2\left(D_{-}^{x}\right)^{ab} \lambda_k^b \\ &\approx 0 \;, \end{aligned} \tag{12}$$

The relation (32) is a secondary constraint, the Gauss's law.

The consistency condition of the secondary constraint yields,

$$\dot{G}_a = \{G_a(x), H_P\} \approx 0. \tag{13}$$

then the Gauss's law is automatically conserved.

The set of first class constraints is $\{\pi_a^+ \,,\ G_a\}$ and the set of second class constraints is $\left\{\phi_a^k\right\}$ whose PB's are

$$\left\{\phi_a^k(x), \phi_b^l(y)\right\} = -2\delta_k^l \left(D_-^x\right)^{ab} \delta^3(x-y) \tag{14}$$

Equation of motion

The time evolution of the fields is determined by computing their PB's with the so called extended hamiltonian H_E,

$$\begin{aligned} \mathrm{H}_E \;=\; & \int d^3y \left\{ \frac{1}{2}\left(\pi_b^-\right)^2 + \pi_b^- \left(D_-^y\right)^{bc} A_+^c + \pi_b^i \left(D_i^y\right)^{bc} A_+^c + \frac{1}{4}\left(F_{ij}^b\right)^2 \right. \\ & \left. + \lambda_l^b \phi_b^l + \mathrm{u}^b \phi_b + \mathrm{v}^b G_b \right\} \end{aligned} \tag{15}$$

thus, we have the time evolution of the fields:

$$\dot{A}_+^a = \mathrm{u}^a, \tag{16}$$

$$\dot{A}_-^a = \pi_a^- + \left(D_-^x\right)^{ac} A_+^c - \left(D_-^x\right)^{ab} \mathrm{v}^b, \tag{17}$$

$$\dot{A}_k^a = \left(D_k^x\right)^{ac} A_+^c + \lambda_k^a - \left(D_k^x\right)^{ab} \mathrm{v}^b, \tag{18}$$

and the canonical momenta,

$$\begin{aligned} \dot{\pi}_a^{+} &= G_a, \\ \dot{\pi}_a^{-} &= -g\varepsilon_{abc}\pi_b^{-}A_+^c + \left(D_l^x\right)^{ab}\lambda_l^b - g\varepsilon_{bca}\mathrm{v}^b\pi_c^{-}, \\ \dot{\pi}_a^{k} &= -g\varepsilon_{bca}\pi_b^k A_+^c + \left(D_j^x\right)^{ab}F_{kj}^b - \left(D_-^x\right)^{ab}\lambda_k^b - g\varepsilon_{abc}\pi_c^k\mathrm{v}^b. \end{aligned} \tag{19}$$

The Dirac's algorithm requires as many gauge conditions as first class constraints there are, thus, we choose the the null-plane gauge

$$A_-^a \approx 0 \qquad , \qquad \pi_a^{-} + \partial_-^x A_+^a \approx 0. \tag{20}$$

as our gauge conditions on the null-plane.

Dirac Brackets

The prescription for determining the Dirac's brackets implies in calculating the inverse of the second class matrix, if we rename the second class constraints as

$$\begin{aligned}
\Theta_1 &\equiv \pi_a^{+} \quad , \quad \Theta_2 \equiv \left(D_{-}^{x}\right)^{ab}\pi_b^{-} + \left(D_i^{x}\right)^{ab}\pi_b^{i} \\
\Theta_3 &\equiv A_{-}^{a} \quad , \quad \Theta_4 \equiv \pi_a^{-} + \partial_{-}^{x} A_{+}^{a} \\
\Theta_5 &\equiv \pi_a^{k} - \partial_{-}^{x} A_k^{a} + \partial_k^{x} A_{-}^{a} - g\varepsilon_{abc}A_{-}^{b}A_k^{c},
\end{aligned} \tag{21}$$

and we define the elements of the second class matrix as $F_{ab}\left(x,y\right) \equiv \left\{\Theta_a\left(x\right), \Theta_b\left(y\right)\right\}$.

The Dirac's bracket of two dynamical variables, $\mathbf{A}_a(x)$ and $\mathbf{B}_b(y)$, is then defined as:

$$\left\{\mathbf{A}_a(x), \mathbf{B}_b(y)\right\}_D = \left\{\mathbf{A}_a(x), \mathbf{B}_b(y)\right\} - \int d^3u d^3v \left\{\mathbf{A}_a(x), \Theta_c(u)\right\} \left(F^{-1}\right)^{cd}(u, v) \left\{\Theta_d(v), \mathbf{B}_b(y)\right\}, \tag{22}$$

where F^{-1} is the inverse of the constraint matrix.

From (22) we obtain the DB among the independent variables of the theory

$$\begin{aligned} \left\{A_k^a(x), A_l^b(y)\right\}_D &= -\frac{1}{4}\delta_b^a\delta_k^l\epsilon(\,x-y)\,\delta^2(x^\intercal - y^\intercal) \\ \left\{A_k^a(x), A_+^b(y)\right\}_D &= \frac{1}{4}|\,x-y|\,(D_k^x)^{ab}\,\delta^2(x^\intercal - y^\intercal)\,. \end{aligned} \tag{23}$$

At once, via the correspondence principle we obtain the commutators among the fields

$$\left[A_k^a(x), A_l^b(y)\right] = -\frac{i}{4}\delta_b^a\delta_k^l\epsilon(\,x-y)\,\delta^2(x^\intercal - y^\intercal)\,, \tag{24}$$

$$\left[A_k^a(x), A_+^b(y)\right] = \frac{i}{4}|\,x-y|\,(D_k^x)^{ab}\,\delta^2(x^\intercal - y^\intercal)\,. \tag{25}$$

Functional Quantization

The theory is characterized by a set of primary constraints:

$$\begin{aligned} \phi_a &\equiv \pi_a^+ \approx 0, & (26) \\ \phi_a^k &\equiv \pi_a^k - \partial_- A_k^a + \partial_k A_-^a - g\varepsilon_{abc} A_-^b A_k^c \approx 0. & (27) \end{aligned}$$

The consistent of primary constraint (26) and (27)) implies the secondary constraint

$$G_a \equiv (D_-)^{ab}\,\pi_b^- + (D_i)^{ab}\,\pi_b^i \approx 0, \tag{28}$$

the Gauss's law. The equation (26) and (28) together constitute a set of first class constraints Ψ_i^a, while (27) is a second class constraint.

For each first class constraint, Ψ_i^a, it is necessary to introduce a gauge condition Δ_j^a, the restriction on our choice of Δ_j^a is that $\det\left|\left\{\Psi_i^a(x), \Delta_j^b(y)\right\}\right| \neq 0$. We choose as the first gauge condition

$$\Delta_1^a \equiv A_-^a \approx 0. \tag{29}$$

The relation (29) will hold for all time only if

$$\Delta_2^a \equiv \pi_a^- + \left(D_-^x\right)^{ab} A_+^b \approx 0. \tag{30}$$

Therefore, the equation (29) and (30) constitute our gauge conditions on the null-plane.

Now, the expression for the transition amplitude of the Yang Mills theory on the null-plane gauge can be written in the following way

$$Z = \int D\mu \ \exp\left\{ i \int d^4x \left(\pi^{\mu}_{a} \partial_{+} A^{a}_{\mu} - \mathcal{H}_c \right) \right\}. \qquad (31)$$

The integration measure is defined by

$$\begin{aligned} D\mu &= D\pi^{\mu}_{a} D A^{a}_{\mu} \ \det \left| \left\{ \Psi^{a}_{i}(x), \Delta^{b}_{j}(y) \right\} \right| \left| \det \left\{ \phi^{k}_{a}(x), \phi^{l}_{b}(y) \right\} \right|^{\frac{1}{2}} \\ & \delta\left(\Psi^{a}_{i}\right) \delta\left(\Delta^{a}_{j}\right) \delta\left(\phi^{k}_{a}\right). \end{aligned} \qquad (32)$$

Here, $\det\left|\left\{\Psi_i^a(x),\Delta_j^b(y)\right\}\right|$ represents the determinant formed by the brackets between the first class constraints and the gauge fixing conditions, which takes the form

$$\det\left|\left\{\Psi_i^a(x),\Delta_j^b(y)\right\}\right|=\delta_b^a\det\left|\begin{pmatrix}0 & 0 & 0 & \partial_-^x\\ 0 & 0 & -\partial_-^x & 0\\ 0 & -\partial_-^x & 0 & 1\\ \partial_-^x & 0 & -1 & 0\end{pmatrix}\delta^3(x-y)\right|, \tag{33}$$

thus, it does not contains field variables and can be absorbed in a normalization constant.

Meanwhile, $\left|\det\left\{\phi_a^k(x),\phi_b^l(y)\right\}\right|^{\frac{1}{2}}$ is the determinant associated to the second class constraints with

$$\left\{\phi_a^k(x),\phi_a^l(y)\right\} = -2\delta_l^k\delta_b^a\partial_-^x\delta^3(x-y). \tag{34}$$

We arrived in the following expression for the transition amplitude

$$Z = N\int DA_\mu^a\,\delta\left[(D_\mu)^{ab}\partial_- A^{b\ \mu}\right]\delta\left(A_-^a\right)\exp\left\{i\int d^4x\left[-\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a\right]\right\}, \tag{35}$$

with

$$N \equiv \det\left|\left\{\Psi_i^a(x),\Delta_j^b(y)\right\}\right|\left|\det\left\{\phi^{a\ k}(x),\phi^{b\ l}(y)\right\}\right|^{\frac{1}{2}}. \tag{36}$$

If we introduce a space-like constant vector

$$n_a^\mu = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}, \tag{37}$$

we can write $(D_\mu)^{ab} \, \partial_- A^{b\ \mu} = (D_\mu)^{ab} \left(n_a^\mu \partial_\mu A^{b\ \mu} \right)$ and

$$A_-^a = n_a^\mu A_\mu^a = n \cdot A \approx 0. \tag{38}$$

The transition amplitude can be written in the form

$$\begin{aligned} Z &= N \int D A_\mu^a \, \delta \left[(D_\mu)^{ab} \left(n_a^\mu \partial_\mu A^{b\ \mu} \right) \right] \delta \left(n \cdot A \right) \\ & \exp \left\{ i \int d^4x \left[-\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a \right] \right\}. \end{aligned} \tag{39}$$

The relation (35) can be expressed in terms of its freedom degree, thus, it can be write

$$
\begin{aligned}
Z &= N \int DA_{+}^{a} DA_{-}^{a} DA_{i}^{a} \\
& \delta\left[(D_{+})^{ab}\,\partial_{-}A^{b\ +} + (D_{-})^{ab}\,\partial_{-}A^{b\ -} + (D_{i})^{ab}\,\partial_{-}A^{b\ i}\right]\delta\left(A_{-}^{a}\right) \\
& \exp\left\{i\int d^{4}x\left[\frac{1}{2}\left(F_{+-}^{a}\right)^{2} + F_{-i}^{a}F_{+i}^{a} - \frac{1}{4}\left(F_{ij}^{a}\right)^{2}\right]\right\}.
\end{aligned}
\tag{40}
$$

Integrating with respect to A_-^a and A_+^a finally we obtain

$$
\begin{aligned}
Z \;=\; & \frac{N}{\det \partial_-} \int DA_i^a \; \exp \left\{ i \int d^4x \left[-\frac{1}{2} A_i \cdot \Box A_i \right.\right. \\
& + g \left[(A_i \times A_j) \cdot \partial_i A_j + \partial_j A_j \cdot \frac{1}{\partial_-} (A_i \times \partial_- A_i) \right] \\
& - g^2 \left[\frac{1}{4} (A_i \times A_j) \cdot (A_i \times A_j) \right. \\
& \left.\left.\left. + \frac{1}{2} \frac{1}{\partial_-} (A_i \times \partial_- A_i) \cdot \frac{1}{\partial_-} (A_j \times \partial_- A_j) \right] \right] \right\} ,
\end{aligned}
\tag{41}
$$

where the "dot" and "cross" products are defined over the internal group space.

Conclusions

- Performing a careful analysis of the constraint structure of Yang-Mills field, we have determined in addition of the usual first class constraints set a second class ones set, which is a characteristic of the null-plane dynamics.
- The imposition of appropriated boundary conditions on the fields fixes the hidden subset of first class constraints and eliminates the ambiguity on the operator ∂_-, that allows to get a unique inverse for the second class constraint matrix.
- The Dirac's brackets of the theory are quantized via correspondence principle; the commutators obtained are the same derived by Tomboulis.

- The relation $A^a_\mu \approx 0$ alone does not define the null-plane gauge. The subsidiary condition $(D_\mu)^{ab}\,\partial_- A^{b\ \mu} \approx 0$ is necessary in order to fix the first-class constraints of the theory.
- Appropriate boundary conditions were imposed on the fields and then the transition amplitudes expressed in terms of the physical components.
- The results found are thoroughly consistent with the ones reported in the literature